Approximation of mixed order Sobolev functions on the d-torus – Asymptotics, preasymptotics and d-dependence

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Dedicated to Vladimir N. Temlyakov on the occasion of his 60th birthday

January 6, 2014

Abstract

We investigate the approximation of d-variate periodic functions in Sobolev spaces of dominating mixed (fractional) smoothness $s > 0$ on the d-dimensional torus, where the approximation error is measured in the L_2-norm. In other words, we study the approximation numbers of the Sobolev embeddings $H^s_{\text{mix}}(\mathbb{T}^d) \hookrightarrow L_2(\mathbb{T}^d)$, with particular emphasis on the dependence on the dimension d. For any fixed smoothness $s > 0$, we find the exact asymptotic behavior of the constants as $d \to \infty$. We observe super-exponential decay of the constants in d, if n, the number of linear samples of f, is large. In addition, motivated by numerical implementation issues, we also focus on the error decay that can be achieved by low rank approximations. We present some surprising results for the so-called "preasymptotic" decay and point out connections to the recently introduced notion of quasi-polynomial tractability of approximation problems.

Keywords Approximation numbers · Sobolev spaces of mixed smoothness · rate of convergence · preasymptotics · d-dependence · quasi-polynomial tractability

Mathematics Subject Classifications (2000) 42A10; 41A25; 41A63; 46E35; 65D15

1 Introduction

In the present paper we investigate the behavior of the approximation numbers of the embeddings

$$I_d : H^s_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\,, \quad s > 0\,, \quad d \in \mathbb{N}\,,$$

where $H^s_{\text{mix}}(\mathbb{T}^d)$ is the periodic Sobolev space of dominating mixed fractional smoothness s on the d-torus $\mathbb{T}^d$ represented in $\mathbb{R}^d$ by the cube $[0, 2\pi]^d$. The approximation numbers of a bounded linear operator $T : X \to Y$ between two Banach spaces are defined as

$$\begin{aligned} a_n(T:\, X \to Y) &:= \inf_{\text{rank}\, A<n} \sup_{\|x|X\|\le 1} \|\, Tx - Ax|Y\| \\ &= \inf_{\text{rank}\, A<n} \|T - A : X \to Y\|, \qquad n \in \mathbb{N}\,. \end{aligned} \tag{1.1}$$

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They describe the best approximation of T by finite rank operators. If X and Y are Hilbert spaces and T is compact, then $a_n(T)$ is the nth singular number of T.

The first result on the approximation of Sobolev embeddings is due to Kolmogorov [14]. He showed already in 1936 that in the univariate (homogeneous) case with integer smoothness $m \in \mathbb{N}$ the approximation numbers $a_n(I_d : \dot{H}^m(\mathbb{T}) \to L_2(\mathbb{T}))$ are given by n^{-m}. Here we are interested in the multivariate (inhomogeneous) situation, where d is large, and investigate the approximation numbers $a_n(I_d : H^s_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$ for arbitrary smoothness parameters $s > 0$. The spaces $H^s_{\text{mix}}(\mathbb{T}^d)$ are much smaller than the isotropic spaces $H^s(\mathbb{T}^d)$, and often they are considered as a reasonable model for reducing the computational effort in high-dimensional approximation.

In recent years there has been an increasing interest in the approximation of multivariate functions, since many problems, e.g. in finance or quantum chemistry, are modeled in associated function spaces on high-dimensional domains. It has been shown that the functions which have to be approximated often possess a mixed Sobolev regularity, as for instance eigenfunctions of certain Hamilton operators in quantum chemistry, see Yserentant's lecture note [30] and the references given there.

In [29, Theorem III.4.4] the two-sided estimate

$$c_s(d)\, n^{-s}(\ln n)^{(d-1)s} \le a_n(I_d :\ H^s_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le C_s(d)\, n^{-s}(\ln n)^{(d-1)s}\,, \tag{1.2}$$

for $n \in \mathbb{N}$ can be found. Here the constants $c_s(d)$ and $C_s(d)$, depending only on d and s, were not explicitly determined. Some more references and comments to the history of (1.2) will be given in Subsection 4.5. Our main focus is to clarify, for arbitrary but fixed $s > 0$, the dependence of these constants on d. In fact, it is necessary to fix the norms on the spaces $H^s_{\text{mix}}(\mathbb{T}^d)$ in advance, since the constants $c_s(d)$ and $C_s(d)$ in (1.2) depend on the size of the respective unit balls. Surprisingly, for a collection of quite natural norms (see the next subsection for details) it turns out that we can choose

$$C_s(d) = \Big[\frac{\lambda^d}{(d-1)!}\Big]^s\,,$$

with $2 \le \lambda \le 6$ depending on the chosen norm. Note that $C_s(d)$ decays super-exponentially in d. This observation can be compared to similar results in Bungartz, Griebel [4], Griebel [9], Schwab et al. [23] and Dũng, Ullrich [7], where the authors noticed at least exponential decay of the constants. A more detailed comparison will be made in Subsection 4.5.

Let us ignore the constants $c_s(d)$ and $C_s(d)$ for a moment, and fix $s > 0$. Then, for arbitrary $d \in \mathbb{N}$, the function $f_d(t) := t^{-s}\,(\ln t)^{s(d-1)}$ is increasing on $[1, e^{d-1}]$ and decreasing on $[e^{d-1}, \infty)$, hence its maximum on $[1,\infty)$ is

$$\max_{t\ge 1} f_d(t) = f_d(e^{d-1}) = \Big(\frac{d-1}{e}\Big)^{s(d-1)}\,,$$

which increases super-exponentially in d. That means, for large d we have to wait "exponentially long" until the sequence $n^{-s}(\ln n)^{(d-1)s}$ decays, and even longer until it becomes less than one. Note that for all norms on $H^s_{\text{mix}}(\mathbb{T}^d)$ to be considered in this paper, we have $a_n(I_d :\ H^s_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le 1$ for all n. Consequently, for small values of n the behavior of $a_n(I_d :\ H^s_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$ is not properly reflected by the asymptotic rate $n^{-s}(\ln n)^{(d-1)s}$.

This is the reason why we split our investigations into three parts. First, we show that the limit

$$\lim_{n\to\infty} \frac{n^s \cdot a_n(I_d : H^s_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))}{(\ln n)^{(d-1)s}} = \Big[\frac{2^d}{(d-1)!}\Big]^s$$

exists, having the same value for various norms. Secondly, for exponentially large n, we calculate some admissible constants $c_s(d)$ and $C_s(d)$. Finally, we consider in some detail the situation of small n, more precisely in the range $1 \le n \le 4^d$. For large d this is the most interesting part for practical issues, since 4^d pieces of information might already be too much for any reasonable algorithm.

The paper is organized as follows. In Section 2 we introduce and investigate the Sobolev spaces of interest. Here we are mainly interested in some assertions on equivalent norms and embeddings. In Subsection 2.2 we add a few remarks to isotropic Sobolev spaces and their relation to Sobolev spaces of dominating mixed smoothness. Subsection 2.3 in this section is devoted to some basics on approximation numbers, in particular, in connection with diagonal operators. In Section 3 we study some combinatorial identities and estimates. Section 4 contains our main results. The final Section 5 transfers our approximation results into the recently introduced notion of quasi-poynomial tractability of the respective approximation problems.

Notation. As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0$ the non-negative integers, $\mathbb{Z}$ the integers and $\mathbb{R}$ the real numbers. With $\mathbb{T}$ we denote the torus represented by the interval $[0, 2\pi]$. For a real number a we put $a_+ := \max\{a, 0\}$ and denote by $[a]$ its greatest integer part. The letter d is always reserved for the dimension in $\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{N}^d$, and $\mathbb{T}^d$. For $0 < p \le \infty$ and $x \in \mathbb{R}^d$ we denote $|x|_p = (\sum_{i=1}^d |x_i|^p)^{1/p}$ with the usual modification for $p = \infty$. The symbol $\#\Omega$ stands for the cardinality of the set Ω. If X and Y are two Banach spaces, the norm of an element x in X will be denoted by $\|x|X\|$ and the norm of an operator $A : X \to Y$ by $\|A : X \to Y\|$. The symbol $X \hookrightarrow Y$ indicates that there is a continuous embedding from X into Y. The equivalence $a_n \sim b_n$ means that there are constants $0 < c_1 \le c_2 < \infty$ such that $c_1 a_n \le b_n \le c_2 a_n$ for all $n \in \mathbb{N}$.

2 Preliminaries

2.1 Sobolev spaces of dominating mixed smoothness on the d-torus

All results in this paper are stated for function spaces on the d-torus $\mathbb{T}^d$, which is represented in the Euclidean space $\mathbb{R}^d$ by the cube $\mathbb{T}^d = [0, 2\pi]^d$, where opposite sides are identified. In particular, for functions f on $\mathbb{T}$, we have $f(x) = f(y)$ whenever $x - y = 2\pi k$ for some $k \in \mathbb{Z}^d$. These functions can be viewed as 2π-periodic in each component.

The space $L_2(\mathbb{T}^d)$ consists of all (equivalence classes of) measurable functions f on $\mathbb{T}^d$ such that norm

$$\|f|L_2(\mathbb{T}^d)\| := \Big(\int_{\mathbb{T}^d} |f(x)|^2\, dx\Big)^{1/2}$$

is finite. All information on a function $f \in L_2(\mathbb{T}^d)$ is encoded in the sequence $(c_k(f))_k$ of its Fourier coefficients, given by

$$c_k(f) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} f(x)\, e^{-ikx}\, dx\,, \qquad k \in \mathbb{Z}^d\,.$$

Indeed, we have Parseval's identity

$$\|f|L_2(\mathbb{T}^d)\|^2 = \sum_{k\in\mathbb{Z}^d} |c_k(f)|^2 \tag{2.1}$$

as well as

$$f(x) = \frac{1}{(2\pi)^{d/2}} \sum_{k\in\mathbb{Z}^d} c_k(f)\, e^{ikx}$$

with convergence in $L_2(\mathbb{T}^d)$.

The mixed Sobolev space $H^m_{\text{mix}}(\mathbb{T}^d)$ of integer smoothness $m \in \mathbb{N}$ is the collection of all $f \in L_2(\mathbb{T}^d)$ such that all distributional derivatives $D^\alpha f$ of order $\alpha = (\alpha_1, ..., \alpha_d)$ with $\alpha_j \le m$, $j = 1, ..., d$, belong to $L_2(\mathbb{T}^d)$. We put

$$\|\, f\,|H^m_{\text{mix}}(\mathbb{T}^d)\| := \Big(\sum_{|\alpha|_\infty \le m} \|\, D^\alpha f\,|L_2(\mathbb{T}^d)\|^2 \Big)^{1/2} . \tag{2.2}$$

One can rewrite this definition in terms of Fourier coefficients. Taking $c_k(D^\alpha f) = (ik)^\alpha c_k(f)$ into account, Parseval's identity (2.1) implies (using the convention $0^0 = 1$)

$$\begin{aligned} \|\, f\,|H^m_{\text{mix}}(\mathbb{T}^d)\|^2 &= \sum_{|\alpha|_\infty \le m} \Big\| \frac{1}{(2\pi)^{d/2}} \sum_{k\in\mathbb{Z}^d} c_k(f)\,(ik)^\alpha e^{ikx} \Big| L_2(\mathbb{T}^d) \Big\|^2 \\ &= \sum_{k\in\mathbb{Z}^d} |c_k(f)|^2 \Big(\sum_{|\alpha|_\infty\le m} \prod_{j=1}^d |k_j|^{2\alpha_j} \Big) = \sum_{k\in\mathbb{Z}^d} |c_k(f)|^2 \Big(\prod_{j=1}^d \sum_{\alpha=0}^m |k_j|^{2\alpha} \Big) \\ &= \sum_{k\in\mathbb{Z}^d} |c_k(f)|^2 \Big(\prod_{j=1}^d \sum_{\alpha=0}^m |k_j|^{2\alpha} \Big) = \sum_{k\in\mathbb{Z}^d} |c_k(f)|^2 \prod_{j=1}^d v_m(k_j)^2 , \end{aligned} \tag{2.3}$$

where

$$v_m(\ell)^2 = \sum_{\alpha=0}^m |\ell|^{2\alpha} \tag{2.4}$$

Due to our convention $0^0 = 1$ we have $v_m(0) = 1$, moreover $v_m(\pm 1) = m + 1$. Defining

$$w_m(k) := \prod_{j=1}^d v_m(k_j) \qquad \text{for } k = (k_1, ..., k_d) \in \mathbb{Z}^d , \tag{2.5}$$

we obtain

$$\|\, f\,|H^m_{\text{mix}}(\mathbb{T}^d)\| = \Big[\sum_{k\in\mathbb{Z}^d} |c_k(f)|^2 w_m(k)^2 \Big]^{1/2} .$$

We could have also started with the equivalent norm

$$\|\, f\,|H^m_{\text{mix}}(\mathbb{T}^d)\|^* := \Big(\sum_{\alpha\in\{0,m\}^d} \|\, D^\alpha f\,|L_2(\mathbb{T}^d)\|^2 \Big)^{1/2} . \tag{2.6}$$

Similarly as above, a reformulation of (2.6) in terms of Fourier coefficients yields

$$\|\, f\,|H^m_{\text{mix}}(\mathbb{T}^d)\|^* = \Big[\sum_{k\in\mathbb{Z}^d} |c_k(f)|^2 \prod_{j=1}^d (1 + |k_j|^{2m}) \Big]^{1/2} . \tag{2.7}$$

Inspired by (2.3) and (2.7) we define Sobolev spaces of dominating mixed smoothness of fractional order $s > 0$ as follows.

Definition 2.1. *Let $s > 0$. The periodic Sobolev space $H^s_{\text{mix}}(\mathbb{T}^d)$ is the collection of all $f \in L_2(\mathbb{T}^d)$ such that $\|f|H^s_{\text{mix}}(\mathbb{T}^d)\| < \infty$, where*
(i) *the classical (natural) norm $\|f|H^s_{\text{mix}}(\mathbb{T}^d)\|^+$ is defined as*

$$\| f \,|H^s_{\text{mix}}(\mathbb{T}^d)\|^+ := \Big[\sum_{k\in\mathbb{Z}^d} |c_k(f)|^2 \prod_{j=1}^d \big(1 + |k_j|^2\big)^s\Big]^{1/2} , \tag{2.8}$$

(ii) *the modified classical norm $\|f|H^s_{\text{mix}}(\mathbb{T}^d)\|^*$ is defined as*

$$\| f \,|H^s_{\text{mix}}(\mathbb{T}^d)\|^* := \Big[\sum_{k\in\mathbb{Z}^d} |c_k(f)|^2 \prod_{j=1}^d \big(1 + |k_j|^{2\,s}\big)\Big]^{1/2} , \tag{2.9}$$

(iii) *and the norm $\|f|H^s_{\text{mix}}(\mathbb{T}^d)\|^\#$ is a further modification defined by*

$$\| f \,|H^s_{\text{mix}}(\mathbb{T}^d)\|^\# := \Big[\sum_{k\in\mathbb{Z}^d} |c_k(f)|^2 \prod_{j=1}^d \big(1 + |k_j|\big)^{2s}\Big]^{1/2} .$$

In the sequel we will often use the notation $H^{s,+}_{\text{mix}}(\mathbb{T}^d)$, $H^{s,*}_{\text{mix}}(\mathbb{T}^d)$, and $H^{s,\#}_{\text{mix}}(\mathbb{T}^d)$ to indicate which of these equivalent norms on $H^s_{\text{mix}}(\mathbb{T}^d)$ we are considering. For integer smoothness $s = m \in \mathbb{N}$ all three norms are also equivalent to the norm given in (2.2). Moreover, in some special cases we do not only have equivalence, but even equality of the norms, namely

$$\| \cdot |H^1_{\text{mix}}(\mathbb{T}^d)\| = \| \cdot |H^1_{\text{mix}}(\mathbb{T}^d)\|^* = \| \cdot |H^1_{\text{mix}}(\mathbb{T}^d)\|^+$$

and

$$\| \cdot |H^{1/2}_{\text{mix}}(\mathbb{T}^d)\|^* = \| \cdot |H^{1/2}_{\text{mix}}(\mathbb{T}^d)\|^\# .$$

Clearly, the size of the unit balls with respect to equivalent norms can be significantly different. Or, in other words, switching from one to another equivalent norm might produce equivalence constants which badly depend on the dimension d. Since we are interested in situations where d is large or even $d \to \infty$, we have to be very careful with these equivalence constants. Therefore, in this context, norm one embeddings are of particular interest and will be very useful. The embeddings given in the next lemma are due the monotonicity of the norms $|\cdot|_p$, where $0 < p < \infty$, except (v), which is a simple consequence of the fact that the square of an integer is larger than its absolute value.

Lemma 2.2. *Let $s > 0$. The following embeddings have norm one.*
(i) *If $s \geq 1$, then*

$$H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \hookrightarrow H^{s,+}_{\text{mix}}(\mathbb{T}^d) \hookrightarrow H^{s,*}_{\text{mix}}(\mathbb{T}^d) ,$$

(ii) *if $1/2 \leq s \leq 1$, then*

$$H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \hookrightarrow H^{s,*}_{\text{mix}}(\mathbb{T}^d) \hookrightarrow H^{s,+}_{\text{mix}}(\mathbb{T}^d) ,$$

(iii) *if $s \leq 1/2$, then*

$$H^{s,*}_{\text{mix}}(\mathbb{T}^d) \hookrightarrow H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \hookrightarrow H^{s,+}_{\text{mix}}(\mathbb{T}^d) ,$$

(iv) *if $s > t$, then*

$$H^{s,+}_{\text{mix}}(\mathbb{T}^d) \hookrightarrow H^{t,+}_{\text{mix}}(\mathbb{T}^d) \quad , \quad H^{s,*}_{\text{mix}}(\mathbb{T}^d) \hookrightarrow H^{t,*}_{\text{mix}}(\mathbb{T}^d) \quad , \quad H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \hookrightarrow H^{t,\#}_{\text{mix}}(\mathbb{T}^d)\,,$$

(v) *and finally,*

$$H^{s,+}_{\text{mix}}(\mathbb{T}^d) \hookrightarrow H^{s/2,\#}_{\text{mix}}(\mathbb{T}^d)\,.$$

We also have embeddings where $H^m_{\text{mix}}(\mathbb{T}^d)$ is involved.

Lemma 2.3. *Let $m \in \mathbb{N}$. Then for all $f \in H^m_{\text{mix}}(\mathbb{T}^d)$ the following chain of inequalities holds.*

$$\| f\,|H^m_{\text{mix}}(\mathbb{T}^d)\|^* \le \| f\,|H^m_{\text{mix}}(\mathbb{T}^d)\| \le \| f\,|H^m_{\text{mix}}(\mathbb{T}^d)\|^+ \le \left(\frac{2^m}{m+1}\right)^{d/2} \| f\,|H^m_{\text{mix}}(\mathbb{T}^d)\| \qquad (2.10)$$

Proof. The first inequality in (2.10) is obvious. The second one is a consequence of (2.3) and (2.4), together with the fact that $v_m(\ell)^2 \le (1+|\ell|^2)^m$ for all $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}$. For the third inequality, it is enough to notice that

$$\|I_d : H^m_{\text{mix}}(\mathbb{T}^d) \to H^{m,+}_{\text{mix}}(\mathbb{T}^d)\|^2 = \sup_{k\in\mathbb{Z}^d} \prod_{j=1}^d \frac{(1+|k_j|^2)^m}{v_m(k_j)} = \left(\sup_{k\in\mathbb{N}} \frac{(1+k^2)^m}{1+k^2+\ldots+k^{2m}}\right)^d$$

and that the function $f(x) = \frac{(1+x)^m}{1+x+\ldots+x^m}$ is decreasing on $[1,\infty)$, hence $f(x) \ge f(1) = \frac{2^m}{m+1}$. ∎

The most convenient norm for our purposes is $\|\cdot\,|H^s_{\text{mix}}(\mathbb{T}^d)\|^\#$. In almost all combinatorial estimates given below we use this specific norm. Afterwards, with some additional effort, the results are carried over to the less convenient but more important norms $\|\cdot\,|H^s_{\text{mix}}(\mathbb{T}^d)\|^+$, $\|\cdot\,|H^s_{\text{mix}}(\mathbb{T}^d)\|^*$ and $\|\cdot\,|H^m_{\text{mix}}(\mathbb{T}^d)\|$.

2.2 Isotropic Sobolev spaces on the d-torus

Let $m \in \mathbb{N}$. Then the isotropic Sobolev space $H^m(\mathbb{T}^d)$ is the collection of all $f \in L_2(\mathbb{T}^d)$ such that all distributional derivatives $D^\alpha f$ up to order m belong to $L_2(\mathbb{T}^d)$, i.e.,

$$\| f\,|H^m(\mathbb{T}^d)\| := \Big(\sum_{|\alpha|_1\le m} \| D^\alpha f\,|L_2(\mathbb{T}^d)\|^2\Big)^{1/2} < \infty\,.$$

Fractional versions for $s > 0$ can be introduced by using Fourier coefficients and the norm

$$\| f\,|H^s(\mathbb{T}^d)\|^+ := \Big(\sum_{k\in\mathbb{Z}^d} |c_k(f)|^2 (1+\sum_{j=1}^d |k_j|^2)^s\Big)^{1/2}\,.$$

Based on these norms it is easy to compare the isotropic Sobolev spaces with the Sobolev spaces of dominating mixed smoothness.

Lemma 2.4. *Let $s > 0$. Then we have the chain of continuous embeddings*

$$H^{sd}(\mathbb{T}^d) \hookrightarrow H^s_{\text{mix}}(\mathbb{T}^d) \hookrightarrow H^s(\mathbb{T}^d)\,, \qquad (2.11)$$

and this is best possible, i.e., for all $\varepsilon > 0$,

$$H^{sd-\varepsilon}(\mathbb{T}^d) \not\subset H^s_{\text{mix}}(\mathbb{T}^d) \not\subset H^{s+\varepsilon}(\mathbb{T}^d)\,.$$

Proof. The proof is elementary, so we will omit the details. However, it is of certain interest to note that the embedding operators in (2.11) are always of norm one, i.e.

$$\|I_d : H^{sd,+}(\mathbb{T}^d) \to H^{s,+}_{\text{mix}}(\mathbb{T}^d)\| = \|I_d : H^{s,+}_{\text{mix}}(\mathbb{T}^d) \to H^{s,+}(\mathbb{T}^d)\| = 1$$

for all $s > 0$, and

$$\|I_d : H^{md}(\mathbb{T}^d) \to H^{m}_{\text{mix}}(\mathbb{T}^d)\| = \|I_d : H^{m}_{\text{mix}}(\mathbb{T}^d) \to H^{m}(\mathbb{T}^d)\| = 1$$

for all $m \in \mathbb{N}$. ■

The mixed space $H^s_{\text{mix}}(\mathbb{T}^d)$ is much closer to the space on the left-hand side in (2.11) than to the space on the right-hand side. This is indicated by a short look at the behavior of the approximation numbers. It is known, see, e.g., [29, Chapter 2, Theorems 4.1 and 4.2], that

$$a_s(d)\, n^{-s/d} \le a_n(I_d :\, H^s(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le A_s(d)\, n^{-s/d}\,, \qquad n \in \mathbb{N}\,, \tag{2.12}$$

holds for all n with constants $a_s(d)$ and $A_s(d)$, only depending on d and s , and hence

$$a_n(I_d :\, H^{sd}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \sim n^{-s}\,.$$

This coincides up to a logarithmic perturbation with the behavior of $a_n(I_d : H^s_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$, see (1.2). Roughly speaking, the mixed Sobolev spaces $H^s_{\text{mix}}(\mathbb{T}^d)$ are much smaller than their isotropic counterparts $H^s(\mathbb{T}^d)$. The behavior of the associated approximation numbers is almost the same as in the one-dimensional isotropic case $H^s(\mathbb{T})$. ¿From the very beginning this has been a major motivation to consider spaces of dominating mixed smoothness in approximation theory as well as in the field of information based complexity (IBC). We refer to Babenko [1], Mityagin [15] and Smolyak [28] for early contributions in the framework of approximation theory (these references are also of relevance with respect to (1.2)). More recent results may be found in Temlyakov's monograph [29]. The role of the spaces $H^s_{\text{mix}}(\mathbb{T}^d)$ in IBC is summarized in the recent series of books by Novak and Woźniakowski [17, 19, 20]. Observe that in IBC the spaces are sometimes called Korobov spaces, see, e.g., [17, pp. 341].

Remark 2.5. In [12] we gave a proof of (2.12) with explicit constants $a_s(d)$ and $A_s(d)$ for various equivalent norms.

2.3 Approximation numbers

If $\tau = (\tau_n)_{n=1}^\infty$ is a sequence of real numbers with $\tau_1 \ge \tau_2 \ge \ldots \ge 0$, we define the diagonal operator $D_\tau : \ell_2 \to \ell_2$ by $D_\tau(\xi) = (\tau_n \xi_n)_{n=1}^\infty$. Recall the definition of the approximation numbers (1.1) already given in the introduction. The following fact concerning approximation numbers of diagonal operators is well-known, see e.g. König [13, Section 1.b], Pinkus [22, Theorem IV.2.2], and Novak and Woźniakowski [17, Corollary 4.12]. Comments on the history may be found in Pietsch [21, 6.2.1.3].

Lemma 2.6. *Let τ and D_τ be as above. Then*

$$a_n(D_\tau :\, \ell_2 \to \ell_2) = \tau_n\,, \qquad n \in \mathbb{N}\,.$$

Here the index set of ℓ_2 is $\mathbb{N}$. We need a modification for arbitrary countable index sets J. Then the space $\ell_2(J)$ is the collection of all $\xi = (\xi_j)_{j\in J}$ such that the norm

$$\|\xi|\ell_2(J)\| := \Big(\sum_{j\in J} |\xi_j|^2\Big)^{1/2}$$

is finite. Let $w = (w_j)_{j\in J}$ with $w_j > 0$ for all $j \in J$, and assume that for every $\delta > 0$ there are only finitely many $j \in J$ with $w_j \geq \delta$. Then the non-increasing rearrangement $(\tau_n)_{n\in\mathbb{N}}$ of $(w_j)_{j\in J}$ exists, and $\lim_{n\to\infty} \tau_n = 0$. Defining $D_w : \ell_2(J) \to \ell_2(J)$ by $D_w(\xi) = (w_j\xi_j)_{j\in J}$ for $\xi \in \ell_2(J)$, Lemma 2.6 gives

$$a_n(D_w : \ell_2(J) \to \ell_2(J)) = \tau_n \,.$$

The preceding identity is scalable in the following sense.

Lemma 2.7. *Let J be a countable index set, let $w = (w_j)_{j\in J}$ and $(\tau_n)_{n\in\mathbb{N}}$ be as above. Then, setting $w^s = (w_j^s)_{j\in J}$, one has for any $s > 0$*

$$a_n(D_{w^s} : \ell_2(J) \to \ell_2(J)) = a_n(D_w : \ell_2(J) \to \ell_2(J))^s = \tau_n^s \,.$$

Now we can reduce our problem on embedding operators in function spaces to the considerably simpler context of diagonal operators in sequence spaces, where index set is $J = \mathbb{Z}^d$. To this end, we consider the operators

$$A_s : H^{s,+}_{\mathrm{mix}}(\mathbb{T}^d) \to \ell_2(\mathbb{Z}^d) \qquad \text{and} \qquad B_s : \ell_2(\mathbb{Z}^d) \to H^{s,+}_{\mathrm{mix}}(\mathbb{T}^d)$$

defined by

$$A_s f = (w_s^+(k) c_k(f))_{k\in\mathbb{Z}^d} \qquad \text{and} \qquad B_s\xi = (2\pi)^{-d/2} \sum_{k\in\mathbb{Z}^d} \frac{\xi_k}{w_s^+(k)} e^{ikx} \,,$$

where the weights are $w_s^+(k) := \prod_{j=1}^d \big(1 + |k_j|^2\big)^{s/2}$. Note the semigroup property of these weights, i.e., $w_s^+(k) \cdot w_t^+(k) = w_{s+t}^+(k)$. Furthermore, we put for $k \in \mathbb{Z}^d$

$$w(k) := \frac{w_{s_1}^+(k)}{w_{s_0}^+(k)}$$

and make use of the associated diagonal operator D_w. Then the following commutative diagram illustrates the situation quite well in case $s_0 > s_1 \geq 0$:

$$\begin{array}{ccc} H^{s_0,+}_{\mathrm{mix}}(\mathbb{T}^d) & \xrightarrow{\;I_d\;} & H^{s_1,+}_{\mathrm{mix}}(\mathbb{T}^d) \\ \Big\downarrow A_{s_0} & & \Big\uparrow B_{s_1} \\ \ell_2(\mathbb{Z}^d) & \xrightarrow{\;D_w\;} & \ell_2(\mathbb{Z}^d) \end{array}$$

By the definition of the norm $\| \cdot |H^s_{\rm mix}(\mathbb{T}^d)\|^+$ it is clear that A_s and B_s are isometries and $B_s = A_s^{-1}$. For the embedding $I_d : H^{s_0,+}_{\rm mix}(\mathbb{T}^d) \to H^{s_1,+}_{\rm mix}(\mathbb{T}^d)$ if $s_0 > s_1 \geq 0$ we obtain the factorization

$$I_d = B_{s_1} \circ D_w \circ A_{s_0} \,. \tag{2.13}$$

The multiplicativity of the approximation numbers applied to (2.13) implies

$$a_n(I_d) \leq \|A_{s_0}\| a_n(D_w) \|B_{s_1}\| = a_n(D_w) = \tau_n \,,$$

where $(\tau_n)_{n=1}^\infty$ is the non-increasing rearrangement of $(w(k))_{k\in\mathbb{Z}^d}$. The reverse inequality can be shown analogously. This gives the important identity

$$a_n(I_d) = a_n(D_w) = \tau_n \,. \tag{2.14}$$

Of course, (2.14) also holds for $I_d : H^{s_0,\#}_{\rm mix}(\mathbb{T}^d) \to H^{s_1,\#}_{\rm mix}(\mathbb{T}^d)$ and for $I_d : H^{s_0,*}_{\rm mix}(\mathbb{T}^d) \to H^{s_1,*}_{\rm mix}(\mathbb{T}^d)$ with the obvious adaption of the weights. Due to the semigroup property mentioned above and Lemma 2.7 we have in particular the nice properties

$$\begin{aligned} a_n(I_d : H^{s_0,+}_{\rm mix}(\mathbb{T}^d) \to H^{s_1,+}_{\rm mix}(\mathbb{T}^d)) &= a_n(I_d : H^{s_0-s_1,+}_{\rm mix}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \\ &= a_n(I_d : H^{1,+}_{\rm mix}(\mathbb{T}^d)_{\rm mix} \to L_2(\mathbb{T}^d))^{s_0-s_1} \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} a_n(I_d : H^{s_0,\#}_{\rm mix}(\mathbb{T}^d) \to H^{s_1,\#}_{\rm mix}(\mathbb{T}^d)) &= a_n(I_d : H^{s_0-s_1,\#}_{\rm mix}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \\ &= a_n(I_d : H^{1,\#}_{\rm mix}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))^{s_0-s_1} \,. \end{aligned}$$

For the norm $\| \cdot \|^*$ the corresponding weights are

$$w^*_s(k) = \prod_{j=1}^d \big(1 + |k_j|^{2s}\big)^{1/2} \,.$$

Note, that they do not satisfy the semigroup property, whence a counterpart of (2.15) does not hold.

3 Some combinatorics

In most considerations below, a crucial role will be played by the cardinality $C(r,d)$ of the set

$$\mathcal{N}(r,d) := \Big\{ k \in \mathbb{Z}^d : \quad \prod_{i=1}^d (1+|k_j|) \leq r \Big\} \,, \qquad r \in \mathbb{N} \,.$$

Lemma 3.1. *For $r \in \mathbb{N}$ we have*

$$C(r,d) = 1 + \sum_{\ell=1}^{\min\{d,\log_2 r\}} 2^\ell \binom{d}{\ell} A(r,\ell) \,, \tag{3.1}$$

where $A(r,\ell) := \#\mathcal{M}(r,\ell)$ with

$$\mathcal{M}(r,\ell) = \Big\{ k \in \mathbb{N}^\ell : \prod_{j=1}^\ell (1+k_j) \leq r \Big\} \,.$$

Proof. The proof is straightforward. The first summand 1 in (3.1) represents the case $k_1 = \ldots = k_d = 0$. Next we group together those vectors k having exactly ℓ non-zero components. This explains why the summation is running from 1 to $\min\{d, \log_2 r\}$. Of course, we may concentrate on those $k \in \mathbb{Z}^d$ with nonnegative components. Since we have ℓ non-zero components, this leads to the factor 2^ℓ. Finally, the binomial coefficient $\binom{d}{\ell}$ is just the number of subsets of $\{1, \ldots, d\}$ of cardinality ℓ. ■

Later on we need estimates of the quantities $A(r,d)$ for all $r \in \mathbb{N}$. Obviously we have $A(r,d) = 0$ for $1 \le r < 2^d$, and $A(2^d, d) = 1$. We intend to relate the number $A(r,\ell)$ to the ℓ-dimensional Lebesgue measure of the set

$$\mathcal{H}_r^\ell := \Big\{x \in \mathbb{R}^\ell \ : \ x_j \ge 1, j = 1, ..., \ell, \prod_{j=1}^{\ell} x_j \le r\} \subset \mathbb{R}^\ell .$$

Here, arbitrary real numbers $r > 1$ are admitted.

Remark 3.2. Of course, $\mathcal{H}_r^\ell$ is essentially the restriction of the hyperbolic cross with parameter r to the first octant in $\mathbb{R}^\ell$. Knowing the classical approximation strategies with respect to the function spaces $H^s_{\text{mix}}(\mathbb{T}^d)$, it is not a surprise that hyperbolic crosses show up here. For easier reference we concentrate on dyadic hyperbolic crosses. For $m, d \in \mathbb{N}$ let

$$H(m,d) := \Big\{k \in \mathbb{Z}^d : \exists u_1, ..., u_d \in \mathbb{N}_0 \quad \text{s.t.} \quad |k_j| \le 2^{u_j} \text{ and } \sum_{j=1}^{d} u_j = m\Big\} .$$

Denote by

$$S_m f(x) := (2\pi)^{-d/2} \sum_{k \in H(m,d)} c_k(f)\, e^{ikx} , \qquad m \in \mathbb{N} ,$$

the associated sequence of partial sums of the Fourier series. Then $N(m) = \operatorname{rank} S_m \sim m^{d-1}\, 2^m$ and

$$\|I_d - S_m : H^s_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\| \sim a_{N(m)}(I_d : H^s_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \sim N(m)^{-s} (\ln N(m))^{(d-1)s} .$$

Here all constants behind $\sim$ are independent of $m \in \mathbb{N}$, but depending on s and d, see Bugrov [2], Nikol'skaya [16], Temlyakov [29] and [24, 25].

Let us return to (3.1). Our next goal will be two-sided estimates for $A(r,\ell)$. Define the function $v_\ell(r) := \text{vol}_\ell(\mathcal{H}_r^\ell)$.

Lemma 3.3. *Let $\ell, r \in \mathbb{N}$ and $r \ge 2^\ell$. Then we have*
(i)

$$2^\ell v_\ell(r/2^\ell) \le A(r,\ell) \le v_\ell(r) \tag{3.2}$$

and (ii)

$$r\Big(\frac{(\ln r)^{\ell-1}}{(\ell-1)!} - \frac{(\ln r)^{\ell-2}}{(\ell-2)!}\Big) \le v_\ell(r) \le r \frac{(\ln r)^{\ell-1}}{(\ell-1)!} , \quad \ell = 2, 3, ...$$

Moreover, the upper estimate in (ii) holds as well in case $\ell = 1$.

Proof. For $k \in \mathbb{N}^\ell$ put $Q_k := k + [0,1]^\ell$. Then it holds

$$\Big\{x \in \mathbb{R}^\ell \ : \ x_j \geq 2, \prod_{j=1}^{\ell} x_j \leq r\Big\} \subset \bigcup_{k \in \mathcal{M}(r,\ell)} Q_k \subset \Big\{x \in \mathbb{R}^\ell \ : \ x_j \geq 1, \prod_{j=1}^{\ell} x_j \leq r\Big\}\,. \qquad (3.3)$$

Taking vol_ℓ in (3.3) together with a change of variable gives (i).

Let us prove (ii) by induction on ℓ. We first define the function

$$f_\ell(r) := r \frac{(\ln r)^{\ell-1}}{(\ell-1)!}$$

and rewrite (ii) as

$$f_\ell(r) - f_{\ell-1}(r) \leq v_\ell(r) \leq f_\ell(r)\,. \qquad (3.4)$$

We consider the upper bound first. One easily verifies the right-hand side in (3.4) in case $\ell = 1$. For $\ell \geq$ we use the recurrence relation

$$v_{\ell+1}(r) = \int_1^r v_\ell(r/t)\, dt\,,$$

which is a consequence of Fubini's theorem. By a change of variable this can be rewritten as

$$v_{\ell+1}(r) = r \int_1^r v_\ell(s)\, \frac{ds}{s^2}\,. \qquad (3.5)$$

This implies

$$v_{\ell+1}(r) = r \int_1^r v_\ell(s)\, \frac{ds}{s^2} \leq r \int_1^r f_\ell(s) \frac{ds}{s^2} = f_{\ell+1}(r)\,.$$

Indeed, the substitution $u = \ln s$ yields

$$r \int_1^r f_\ell(s) \frac{ds}{s^2} = r \int_1^r \frac{s(\ln s)^{\ell-1}}{(\ell-1)!} \frac{ds}{s^2} = \frac{r}{(\ell-1)!} \int_0^{\ln r} u^{\ell-1}\, du = \frac{(\ln r)^\ell}{\ell!} = f_{\ell+1}(r)\,. \qquad (3.6)$$

For the lower bound we first verify the left-hand side in (3.4) in case $\ell = 2$ by using $v_2(r) = r\ln r - r + 1$. The recurrence relation (3.5) together with the induction hypothesis yields

$$v_{\ell+1}(r) = r \int_1^r v_\ell(s) \frac{ds}{s^2} \geq r \int_1^r \Big(f_\ell(s) - f_{\ell-1}(s)\Big) \frac{ds}{s^2} = f_{\ell+1}(r) - f_\ell(r)\,,$$

where the last identity is a consequence of (3.6) . The proof is complete. ■

Remark 3.4. In the recent preprint [6], Dũng and Chernov considered cardinalities and volumes of hyperbolic cross type sets in $\mathbb{R}^d$ similar to $\mathcal{H}_r^\ell$ above, see for instance (1.9), (1.10), Theorem 4.2, and Corollaries 4.3., 4.4, 4.5. However, for our purpose, i.e. the control of the numbers $C(r,d)$, see (3.1) above, the estimates presented here are more appropriate.

4 Approximation numbers of Sobolev embeddings

In this section we will compute, or at least estimate, the approximation numbers of the embedding $I_d : H^s_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$. The main aim is to prove (1.2) with explicit constants $c_s(d)$ and $C_s(d)$. First we deal with the norm $\| \cdot |H^s_{\mathrm{mix}}(\mathbb{T}^d)\|^{\#}$.

4.1 The approximation numbers $a_n(I_d : H^{s,\#}_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$ for large n

For $s > 0$ we put

$$w_s^\#(k) := \prod_{j=1}^d (1 + |k_j|)^s , \qquad k \in \mathbb{Z}^d .$$

Due to Lemma 2.6 and (2.14) we have

$$a_n(I_d : H^{s,\#}_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \sigma_n , \qquad n \in \mathbb{N} ,$$

where $(\sigma_n)_{n\in\mathbb{N}}$ denotes the non-increasing rearrangement of $(1/w_s^\#(k))_{k\in\mathbb{Z}^d}$. We have

$$\{\sigma_n : n \in \mathbb{N}\} = \{1/w_s^\#(k) : k \in \mathbb{Z}^d\} = \{r^{-s} : r \in \mathbb{N}_0\} ,$$

that means $(\sigma_n)_{n\in\mathbb{N}}$ is a piecewise constant sequence. Recall the notation

$$C(r,d) = \#\Big\{k \in \mathbb{Z}^d : \quad \prod_{i=1}^d (1 + |k_j|) \le r\Big\} \; = \; \#\Big\{k \in \mathbb{Z}^d : \; w_s^\#(k) \le r^s\Big\} .$$

These observations imply the following result.

Lemma 4.1. *Let $s > 0$ and $r \in \mathbb{N}$. Then*

$$a_n(I_d : H^{s,\#}_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = r^{-s} , \qquad \textit{if} \quad C(r-1,d) < n \le C(r,d) .$$

Remark 4.2. Of course, without precise information on the behavior of the quantities $C(r,d)$, Lemma 4.1 is not very useful for practical purposes. But it provides, at least in principle, complete knowledge on the sequence of approximation numbers $a_n(I_d : H^{s,\#}_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$. In particular,

$$\begin{aligned} a_1 = 1 > a_2 = \ldots = a_{2d+1} &= 2^{-s} \\ > a_{2d+2} = \ldots = a_{4d+1} &= 3^{-s} \\ > a_{4d+2} = \ldots = a_{2d^2+4d+1} &= 4^{-s} > \ldots . \end{aligned}$$

Furthermore, for any $n \in \mathbb{N}$, we can easily construct optimal algorithms S_n of rank less than n. If $C(r-1,d) < n \le C(r,d)$, we choose

$$S_n f(x) := \sum_{k\in\mathcal{N}(r-1,d)} c_k(f)\, e^{ikx} .$$

In fact, by this construction we get

$$\sup_{\|f|H^s_{\mathrm{mix}}(\mathbb{T}^d)\|^\# \le 1} \|f - S_n f|L_2(\mathbb{T}^d)\| = r^{-s} = a_n(I_d : H^{s,\#}_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) .$$

In a next step we determine the asymptotic behavior of the approximation numbers as $n \to \infty$, including the exact dependence on the smoothness parameter s and the dimension d. Note that the *existence* of the limit in the following result is not at all obvious a priori.

Theorem 4.3. *Let $s > 0$ and $d \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \frac{n^s\, a_n(I_d : H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))}{(\ln n)^{(d-1)s}} = \Big[\frac{2^d}{(d-1)!}\Big]^s .$$

Proof. Fix $d \in \mathbb{N}$. By Lemma 2.7 it is enough to deal with the case $s = 1$. For simplicity of notation we write $a_n := a_n(I_d : H^{1,\#}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$. We have $a_n = 1/r$ if $r \in \mathbb{N}$ and $C(r-1,d) < n \le C(r,d)$, see Lemma 4.1. Clearly $\lim_{r\to\infty} C(r,d) = \infty$, moreover the sequence $n(\ln n)^{-(d-1)}$ is increasing for $n > e^{d-1}$. Hence we obtain for sufficiently large $r \in \mathbb{N}$ the two-sided inequality

$$\frac{C(r-1,d)}{r(\ln C(r-1,d))^{d-1}} \le \frac{n\, a_n}{(\ln n)^{d-1}} \le \frac{C(r,d)}{r(\ln C(r,d))^{d-1}} . \tag{4.1}$$

By (3.1) and (3.2) we have for $r \ge 2^d$

$$C(r,d) \le 1 + \sum_{\ell=1}^{d} \binom{d}{\ell} 2^\ell v_\ell(r) . \tag{4.2}$$

¿From $C(r,d) \ge C(r,1) = 2r-1 \ge r$ for all $r \in \mathbb{N}$, we get $\ln C(r,d) \ge \ln r$. Taking Lemma 3.3/(ii) into account, we arrive at

$$\frac{C(r,d)}{r(\ln C(r,d))^{d-1}} \le \frac{1 + \sum\limits_{\ell=1}^{d} \binom{d}{\ell} 2^\ell r \frac{(\ln r)^{\ell-1}}{(\ell-1)!}}{r(\ln r)^{d-1}} \xrightarrow[r\to\infty]{} \frac{2^d}{(d-1)!} ,$$

since only the last summand contributes to the limit. Together with (4.1) this gives

$$\limsup_{n\to\infty} \frac{n\, a_n}{(\ln n)^{d-1}} \le \frac{2^d}{(d-1)!} .$$

Now let us pass to the estimate from below. By (3.1) and Lemma 3.3/(i),(ii) we have

$$C(r,d) \ge 2^d\, r\, \frac{(\ln r - d\ln 2)^{d-1}}{(d-1)!} \cdot \Big(1 - \frac{d-1}{\ln r - d\ln 2}\Big) \tag{4.3}$$

for $r \ge 2^d$. Next we need a proper upper estimate for $\ln C(r,d)$. In fact, if $r \ge e^{d-1} \ge e^\ell$ we have

$$\frac{(\ln r)^{\ell-1}}{(\ell-1)!} \le \frac{(\ln r)^d}{d!} .$$

Hence, using Stirling's formula, we can estimate

$$C(r,d) \le 1 + \sum_{\ell=1}^{d} \binom{d}{\ell} 2^\ell\, r\, \frac{(\ln r)^d}{d!} = 3^d\, r\, \frac{(\ln r)^d}{d!} \le r\, \Big(\frac{3e\ln r}{d}\Big)^d . \tag{4.4}$$

This gives, for $r \ge e^{d-1}$,

$$\ln C(r,d) \le \ln r + d\ln\ln r + d\ln(3e/d) \tag{4.5}$$

Now let $C(r,d) < n \le C(r+1,d)$. Then we have $a_n = \frac{1}{r+1}$, and inserting the above inequalities in (4.3) yields

$$
\begin{aligned}
\frac{n a_n}{(\ln r)^{d-1}} &\ge \frac{C(r,d)}{(r+1)(\ln C(r,d))^{d-1}} \\
&\ge \frac{2^d}{(d-1)!} \cdot \frac{r}{r+1} \cdot \left(\frac{\ln r - d\ln 2}{\ln r + d\ln\ln r + d\ln(3e/d)} \right)^{d-1} \cdot \left(1 - \frac{d-1}{\ln r - d\ln 2} \right) \\
&\xrightarrow[r\to\infty]{} \frac{2^d}{(d-1)!} \, .
\end{aligned}
$$

This implies

$$
\liminf_{n\to\infty} \frac{n\, a_n}{(\ln n)^{d-1}} \ge \frac{2^d}{(d-1)!} \, ,
$$

and the proof is complete. ■

Remark 4.4. (i) For $d = 1$, the mixed space $H^{s,\#}_{\text{mix}}(\mathbb{T})$ coincides with the isotropic space $H^{s,\#}(\mathbb{T})$, with equality of the corresponding norms. Then

$$
\lim_{n\to\infty} n^s\, a_n(I_d :\, H^{s,\#}(\mathbb{T}) \to L_2(\mathbb{T})) = 2^s
$$

follows, a result, already proved in [12].
(ii) As a consequence of Stirling's formula we observe that

$$
\Big[\frac{2^d}{(d-1)!}\Big]^s \asymp \Big(\frac{d}{2\pi}\Big)^{s/2} \Big(\frac{2e}{d}\Big)^{ds} \, ,
$$

where $a_n \asymp b_n$ means $\lim_{n\to\infty} a_n/b_n = 1$. This shows a super-exponential decay of the constant.

Being interested in explicit constants $c_s(d), C_s(d)$ in (1.2), we can learn something from Theorem 4.3. Fix $d \in \mathbb{N}$ and $s > 0$. Then for any given $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$
\Big[\frac{2^d}{(d-1)!}\Big]^s - \varepsilon \le \frac{n^s\, a_n(I_d :\, H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))}{(\ln n)^{s(d-1)}} \le \Big[\frac{2^d}{(d-1)!}\Big]^s + \varepsilon \qquad \text{for all } n \ge n_0 \, .
$$

Equivalently, for any given $n_1 \in \mathbb{N}$ there is a constant $\lambda = \lambda(n_1)$, $1 < \lambda < \infty$, such that

$$
\frac{1}{\lambda}\Big[\frac{2^d}{(d-1)!}\Big]^s \le \frac{n^s\, a_n(I_d :\, H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))}{(\ln n)^{s(d-1)}} \le \lambda \Big[\frac{2^d}{(d-1)!}\Big]^s \qquad \text{for all } n \ge n_1 \, .
$$

We now aim at controlling the constant $\lambda(n_1)$ for certain (large) values of n_1.

Theorem 4.5. *Let $s > 0$ and $d \in \mathbb{N}$.*
(i) Then

$$
a_n(I_d :\, H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le \Big[\frac{3^d}{(d-1)!}\Big]^s \frac{(\ln n)^{(d-1)s}}{n^s} \qquad \textit{if} \quad n \ge 27^d \, .
$$

(ii) On the other hand,

$$
a_n(I_d :\, H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \ge \left[\frac{3}{d!} \left(\frac{2}{2+\ln 12}\right)^d\right]^s \frac{(\ln n)^{(d-1)s}}{n^s} \qquad \textit{if} \quad n > (12\, e^2)^d \, .
$$

Proof. Again, it is enough to deal with the case $s = 1$.
For $r \in \mathbb{N}$ and $C(r-1,d) < n \le C(r,d)$ we have $a_n = 1/r$, whence $a_n \le 1$ for all n.

Step 1. Proof of (i). First recall that $C(r,d) \ge C(r,1) \ge r$ (see the previous proof), and that $n/(\ln n)^{d-1}$ is increasing for $n > e^{d-1}$. Similarly as above in (4.4) we have, for all $n > e^{d-1}$,

$$\frac{n\,a_n}{(\ln n)^{d-1}} \le \frac{C(r,d)}{r(\ln C(r,d))^{d-1}} \le \frac{1+\sum_{\ell=1}^{d} 2^\ell \binom{d}{\ell} v_\ell(r)}{r(\ln r)^{d-1}} \le \frac{1+\sum_{\ell=1}^{d} 2^\ell \binom{d}{\ell} \frac{(\ln r)^{\ell-1}}{(\ell-1)!}}{(\ln r)^{d-1}}\,.$$

Since $\frac{(\ln r)^{\ell-1}}{(\ell-1)!} \le \frac{(\ln r)^{d-1}}{(d-1)!}$, we have

$$\sup_{n \ge C(e^d,d)} \frac{n\,a_n}{(\ln n)^{d-1}} \le \frac{1}{(d-1)!}\sum_{\ell=0}^{d} 2^\ell \binom{d}{\ell} = \frac{3^d}{(d-1)!}\,.$$

Next we give a precise range for n in which this estimate holds. To this end, we estimate $C(r,d)$ with $r = e^d$ from above. The obvious inequality $x^k/k! \le e^x$ applied to $x = \ln r = d$ gives

$$C(e^d,d) \le r + \sum_{\ell=1}^{d} 2^\ell \binom{d}{\ell} r \frac{(\ln r)^{\ell-1}}{(\ell-1)!} \le 3^d\, r^2 = (3e^2)^d\,. \tag{4.6}$$

This shows that (i) holds for all $n > (3e^2)^d$.

Finally we show that for $n \ge 27^d$ the upper bound in (i) is non-trivial (i.e < 1). To see this, we use $(\ln n)^{d-1}/(d-1)! \le (\ln n)^d/d!$ for $n > e^d$, recall that the function $f_d(t) = t^{-1}\,(\ln t)^{d-1}$ is decreasing on $[e^{d-1},\infty)$. Applying Stirling's formula and these monotonicity assertions, estimate (i) yields

$$a_n \le \frac{(3\ln n)^d}{d!\,n} \le \left(\frac{3ed\ln 27}{27d}\right)^d = \left(\frac{e\ln 3}{3}\right)^d\,.$$

Since $(e\ln 3)/3 = 0.99544... < 1$, we see that indeed $a_n < 1$.

Step 2. Let us turn to the estimate from below. Arguing as in (4.6) we find

$$\ln C(r,d) \le \ln(3^d\, r^2)\,, \qquad r \ge e^d\,. \tag{4.7}$$

Next we estimate of $C(r,d)$ from below. We start with formula (4.3)

$$C(r,d) \ge \frac{2^d\, r}{(d-1)!}\, \Big(\ln(r/2^d)\Big)^{d-1} \left(1 - \frac{d-1}{\ln(r/2^d)}\right).$$

For $r \ge r_0 := (2e)^d$ and $C(r,d) < n \le C(r+1,d)$, using again the monotonicity of f_d, this implies

$$\begin{aligned} \frac{n\,a_n}{(\ln n)^{d-1}} &\ge \frac{C(r,d)}{(r+1)\,(\ln C(r,d))^{d-1}} \\ &\ge \frac{2^d}{(d-1)!}\cdot\frac{r}{r+1}\cdot\left(\frac{\ln(r/2^d)}{\ln(3^d\, r^2)}\right)^{d-1}\cdot\left(1 - \frac{d-1}{\ln(r/2^d)}\right). \end{aligned} \tag{4.8}$$

Concerning the different factors we have, for all $r \geq r_0$,

$$\begin{aligned} \frac{r}{r+1} &\geq \frac{5^d}{5^d+1} \geq \frac{5}{6}, \quad \text{since } 2e > 5, \\ \frac{\ln(r/2^d)}{\ln(3^d\, r^2)} &\geq \frac{\ln(e^d)}{\ln(12^d\, e^{2d})} = \frac{1}{2+\ln 12} \\ 1 - \frac{d-1}{\ln(r/2^d)} &\geq 1 - \frac{d-1}{d} = \frac{1}{d}, \end{aligned}$$

Hence, taking $\frac{5}{6}\,(2+\ln 12) \geq 3$ into account, we arrive at

$$\sup_{n \geq C(r_0,d)} \frac{n\, a_n}{(\ln n)^{d-1}} \geq \frac{3}{d!} \left(\frac{2}{2+\ln 12} \right)^d .$$

Since $C(r_0, d) \leq 3^d\, r_0^2 = (12\, e^2)^d$, the proof of (ii) is finished. ∎

Remark 4.6. (i) We can improve on the bound in (ii), if we choose a larger value of r_0. But then the range of n for which (ii) holds becomes smaller. Here is an example. Taking $r_1 := 48^{d/2} > r_0$, we get

$$\frac{\ln(r_1/2^d)}{\ln(3^d\, r_1^2)} = \frac{1}{4}, \quad \frac{r}{r+1} \geq \frac{\sqrt{48}}{\sqrt{48}+1} \geq \frac{41}{47} \quad \text{and} \quad 1 - \frac{d-1}{\ln(r/2^d)} \geq 1 - \frac{1}{\ln\sqrt{12}} = 0.766173... \geq \frac{3}{4}.$$

Since $\frac{41}{47} \cdot \frac{4}{3} > 1$, we obtain

$$a_n(I_d : H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \geq \left[\frac{3^d (\ln n)^{d-1}}{2^d\, (d-1)!\, n} \right]^s \qquad \text{if} \quad n > (48)^{d/2}.$$

(ii) Conversely, one can extend the range of n in (ii) by making r_0 smaller. However, this strategy is limited by our method. Indeed, if $r \leq 2^d\, e^{d-1}$, then for the last factor in (4.8) we have

$$1 - \frac{d-1}{\ln(r/2^d)} \leq 1 - \frac{d-1}{\ln(e^{d-1})} = 0$$

and our estimate (4.8) becomes useless.

Some “local” improvements

We do not claim that the estimates obtained in Theorem 4.5 are optimal in d and n. They can be improved in various ways. But these improvements take place only locally, i.e., for n taken from a finite interval.

Let $d \in \mathbb{N}$, and let $(\sigma_n)_{n \in \mathbb{N}}$ be the non-increasing rearrangement of $\Big(1/w_s^{\#}(k)\Big)_{k \in \mathbb{Z}^d}$. Now we estimate σ_n by a tensor trick. This method is very simple and works for any $d \in \mathbb{N}$. The best result that can be obtained in this way differs by a log-factor from the exact asymptotic order of σ_n. However, since the resulting constants are quite explicit, it improves on Theorem 4.5, (i) if $15^d < n < \exp(\sqrt{d/(2\pi)} \cdot 1.5^d)$, see Remark 4.8 below.

Lemma 4.7. *For every $d \in \mathbb{N}$, every $s > 0$ and all $n \geq 15^d$ it holds*

$$a_n(I_d : H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq \frac{1}{n^s} \left(\frac{2e \ln n}{d} \right)^{sd} . \tag{4.9}$$

Proof. Again we concentrate on $s = 1$. For arbitrary $p > 1$ we have

$$n\sigma_n^p \le \sum_{j=1}^{n} \sigma_j^p \le \sum_{j=1}^{\infty} \sigma_j^p = \sum_{k\in\mathbb{Z}^d} \prod_{\ell=1}^{d} \frac{1}{(1+|k_\ell|)^p} = \left(\sum_{m\in\mathbb{Z}} \frac{1}{(1+|m|)^p}\right)^d = \left(1 + 2\sum_{m=2}^{\infty} \frac{1}{m^p}\right)^d$$
$$\le \left(1 + 2\int_1^\infty \frac{dx}{x^p}\right)^d = \left(1 + \frac{2}{p-1}\right)^d = \left(\frac{p+1}{p-1}\right)^d ,$$

which implies

$$\sigma_n \le n^{-1/p} \left(\frac{p+1}{p-1}\right)^{d/p} \le n^{-1/p} \left(\frac{p+1}{p-1}\right)^d .$$

Now we optimize, for given $n \in \mathbb{N}$, over the free parameter $p > 1$. Note that the map $p \mapsto \frac{p+1}{p-1}$ is a bijection from the interval $(1, \infty)$ onto itself. If $n > e^{d/2}$, we have $(2 \ln n)/d > 1$, and so we can choose $p > 1$ such that

$$\frac{p+1}{p-1} = \frac{2\ln n}{d} . \tag{4.10}$$

It remains to estimate the exponent in $n^{-1/p}$. We have

$$-\frac{1}{p} = -1 + \frac{p-1}{p+1} \cdot \frac{p+1}{p} \le -1 + \frac{d}{\ln n} ,$$

and hence

$$n^{-1/p} \le n^{-1} \cdot e^d .$$

This implies the desired estimate

$$\sigma_n \le \frac{1}{n} \left(\frac{2e \ln n}{d}\right)^d \qquad \text{for all } n > e^{d/2} .$$

This bound is non-trivial (i.e. < 1) for $n \ge 15^d$. ■

Remark 4.8. (i) In our Theorem 4.5 we got the upper bound (in slightly rewritten form)

$$a_n(I_d : H^{s,\#}_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le \frac{1}{n^s \ln^s n} \cdot \left(\sqrt{\frac{d}{2\pi}} \, \frac{(3e \ln n)^d}{d^d}\right)^s .$$

This bound is larger than the bound obtained in (4.9), if and only if

$$n \le \exp(\sqrt{d/2\pi} \cdot 1.5^d)$$

which is doubly exponential in d, that means far beyond all n in 'real life' applications or in numerical analysis. So the tensor trick might after all be quite useful, although it cannot give the exact asymptotic rate.

(ii) The first part of this remark explains that the choice of p in (4.10) is reasonable, since it almost gives the exact asymptotic rate as $n \to \infty$. However, it is not optimal for all n. This might be seen as follows. We simply fix p from the very beginning and follow the above argument. The most simple choice is $p = 2$. Then we have the exact value of the sum $\sum_{j=1}^\infty \sigma_j^2$ at hand and the outcome is

$$a_n(I_d : H^{s,\#}_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le \left(\frac{1}{n}\left(\frac{\pi^2}{3} - 1\right)^d\right)^{s/2} , \tag{4.11}$$

for all $n \in \mathbb{N}$. Since $2 < \frac{\pi^2}{3} - 1 < e$ this estimate is of certain use if $n \ge e^d$. Now we compare (4.9) and (4.11). It follows

$$\left(\frac{1}{n}\left(\frac{\pi^2}{3}-1\right)^d\right)^{s/2} \le \frac{1}{n^s}\left(\frac{2e\ln n}{d}\right)^{sd}$$

if and only if

$$\frac{n^{1/(2d)}}{\ln n} \le \frac{2e}{d}\left(\frac{\pi^2}{3}-1\right)^{-1/2}.$$

A sufficient condition is given by

$$\frac{n^{1/(2d)}}{\ln n} \le \frac{e}{d}.$$

The function $f(x) := x^{1/(2d)}/\ln x$ is decreasing on $[1, e^{2d}]$ and increasing on $[e^{2d}, \infty)$, and $f(e^d) = \sqrt{e}/d < e/d$. Because of $f(e^{cd}) = (\sqrt{e})^c/(cd) \le e/d$ if and only if $c - 2\ln c \le 2$ we conclude that

$$\left(\frac{1}{n}\left(\frac{\pi^2}{3}-1\right)^d\right)^{s/2} \le \frac{1}{n^s}\left(\frac{2e\ln n}{d}\right)^{sd} \qquad \text{if} \quad e^d \le n \le e^{c_0 d},$$

where c_0 is the solution of $c - 2\ln c = 2$ ($5.35 < c_0 < 5.36$). Hence, (4.11) is better than (4.9) as long as $e^d \le n \le e^{c_0 d}$. Different choices of p (e.g., $p = 3/2$, $p = 4$) lead to different intervals of optimality, we omit further details.

4.2 The approximation numbers $a_n(I_d : H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$

For computational issues, the number $(3e^2)^d$ in Theorem 4.5 might be much too large. We will now focus on estimates for smaller n and investigate the so-called preasymptotic behavior. To be more precise, we will deal with estimates of $a_n(I_d : H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$ from above and below in the range $1 \le n \le (d/2)\, 4^d$.

Theorem 4.9. *Let $s > 0$ and $d \in \mathbb{N}$, $d \ge 2$. For all $1 \le n \le \frac{d}{2}4^d$ it holds*

$$a_n(I_d : H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le \left(\frac{e^2}{n}\right)^{\frac{s}{2+\log_2 d}}. \tag{4.12}$$

Proof. It is enough to consider the case $s = 1$. Let $1 \le r \le 2^d$. Then for $C(r-1, d) < n \le C(r, d)$ we have $a_n = 1/r$. Let us estimate $C(r, d)$ in this case. We shall use $[x]$ to denote the greatest integer part of the real number x. Starting from (3.1) and using the obvious estimate $x^k/k! \le e^x$ applied to $x = \ln r = d$, we obtain

$$\begin{aligned} C(r,d) &= 1 + \sum_{\ell=1}^{[\log_2 r]} 2^\ell \binom{d}{\ell} A(r,\ell) \le 1 + \sum_{\ell=1}^{[\log_2 r]} 2^\ell \binom{d}{\ell} v_\ell(r) \le 1 + \sum_{\ell=1}^{[\log_2 r]} 2^\ell \binom{d}{\ell} r\, \frac{(\ln r)^{\ell-1}}{(\ell-1)!} \\ &\le r^2 \sum_{\ell=0}^{[\log_2 r]} 2^\ell \binom{d}{\ell} \le r^2 \sum_{\ell=0}^{[\log_2 r]} 2^\ell \frac{d^\ell}{\ell!} \le r^2\, d^{[\log_2 r]}\, e^2 \le e^2\, r^{2+\log_2 d}. \end{aligned}$$

This gives $n \le C(r,d) \le e^2\, r^{2+\log_2 d}$ which implies $1/r \le (e^2/n)^{1/(2+\log_2 d)}$. Therefore we get for all $n \le C(r,d)$ the relation

$$a_n \le \left(\frac{e^2}{n}\right)^{\frac{1}{2+\log_2 d}}.$$

This estimate holds for all $n \le C(2^d,d)$. To estimate $C(2^d,d)$ from below we need a preparation. Obviously, in case $\ell \ge 2$, we have

$$\Big\{k \in \mathbb{N}^\ell :\ k_2 = \ \ldots\ = k_\ell = 1\,, \quad (1+k_1)\,2^{\ell-1} \le r\Big\} \subset \mathcal{M}(r,\ell)\,.$$

The set of the left hand side has cardinality $[r2^{-\ell+1}]-1$. By interchanging the roles of k_1 with k_j, $j=2,3,\ldots$, we find ℓ subsets of $\mathcal{M}(r,\ell)$ having only $(1,\ldots,1)$ in the intersection. This implies

$$A(r,\ell) \ge \ell\,[r\,2^{-\ell+1}] - 2\ell + 1\,, \tag{4.13}$$

which is also true for $\ell = 1$. In case $r = 2^d$ we obtain from Lemma 3.1

$$\begin{aligned}
C(2^d,d) &= 1 + \sum_{\ell=1}^{d} 2^\ell \binom{d}{\ell} A(2^d,\ell) \ge 1 + \sum_{\ell=1}^{d} 2^\ell \binom{d}{\ell}\Big(\ell 2^{d-\ell+1} - 2\ell + 1\Big)\\
&= 2^{d+1}\sum_{\ell=1}^{d}\binom{d}{\ell}\,\ell \,-\, 2\sum_{\ell=1}^{d}\binom{d}{\ell}\,\ell\,2^\ell + 3^d\\
&= 3^d + 2^{d+1}\sum_{\ell=1}^{d}\frac{d!}{(d-\ell)!(\ell-1)!} - 2\sum_{\ell=1}^{d}2^\ell\frac{d!}{(d-\ell)!(\ell-1)!}\\
&= 3^d + 2^{d+1}d\sum_{\ell=1}^{d}\binom{d-1}{\ell-1} - 2d\sum_{\ell=1}^{d}2^\ell\binom{d-1}{\ell-1}\\
&= 3^d + 2^{d+1}d2^{d-1} - 4d3^{d-1}\\
&= 3^d + d4^d - 4d3^{d-1}\,.
\end{aligned}$$

Hence, we have $C(2^d,d) \ge 3^d + d4^d - \frac{4}{3}d3^d$. Note, that

$$\begin{aligned}
C(2^d,d) \ge \frac{d}{2}4^d \quad &\Longleftrightarrow \quad 3^d + d4^d - \frac{4}{3}d3^d \ge \frac{d}{2}4^d\\
&\Longleftrightarrow \quad \frac{d}{2}4^d \ge \Big(\frac{4}{3}d - 1\Big)3^d\,.
\end{aligned} \tag{4.14}$$

Of course, (4.14) is true for all $d \ge 2$. The proof is complete. ■

Let us turn to an estimate from below.

Theorem 4.10. *Let $d \in \mathbb{N}$, $d \ge 2$, and $s > 0$. For $n \ge 2$ we define*

$$\alpha(n,d) := 2 + \log_2\Big(\frac{d}{\log_2 n} + \frac{1}{2}\Big)\,. \tag{4.15}$$

For all $2 \le n \le \frac{d}{2}4^d$ it holds

$$a_n(I_d : H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \ge 2^{-s} n^{-\frac{s}{\alpha(n,d)}}\,. \tag{4.16}$$

Proof. It suffices to deal with the case $s=1$. Let $2 \le r \le 2^d$ such that $C(r-1,d) < n \le C(r,d)$. Furthermore, let $m \in \mathbb{N}_0$ be determined from

$$2^m \le r - 1 < 2^{m+1}\,. \tag{4.17}$$

Then (3.1) and (4.13) imply

$$n > C(r-1,d) = 1 + \sum_{\ell=1}^{m} 2^\ell \binom{d}{\ell} A(r,\ell) \ge \sum_{\ell=1}^{m} 2^\ell \binom{d}{\ell} \Big(\ell\, 2^{m-\ell+1} - \ell + 1\Big) \ge 2^{m+1} \sum_{\ell=1}^{m} \binom{d}{\ell}.$$

Hence

$$n > 2^{m+1} \sum_{\ell=1}^{m} \binom{m}{\ell} \frac{d(d-1)\cdot(d-\ell+1)}{m(m-1)\cdots(m-\ell+1)} \ge 2^{m+1} \sum_{\ell=1}^{m} \binom{m}{\ell} \Big(\frac{d}{m}\Big)^\ell.$$

Taking the binomial formula into account, this implies

$$n > 2^{m+1} \Big\{ \Big(1 + \frac{d}{m}\Big)^m - 1 \Big\} \ge 4^m \Big(\frac{d+m}{2m}\Big)^m.$$

Next we apply $\log_2$ on both sides and obtain

$$\log_2 n > m \Big\{ 2 + \log_2 \Big(\frac{d}{2m} + \frac{1}{2}\Big)\Big\} \ge 2m\,, \tag{4.18}$$

since $\frac{d}{2m} + \frac{1}{2} \ge 1$. Together with (4.18) this yields

$$\log_2 n > m \Big\{ 2 + \log_2 \Big(\frac{d}{\log_2 n} + \frac{1}{2}\Big)\Big\} = m \cdot \alpha(n,d)\,.$$

Rewriting this inequality we get

$$2^m < 2^{\frac{\log_2 n}{\alpha(n,d)}} = n^{\frac{1}{\alpha(n,d)}}\,.$$

Taking (4.17) into account, we finally conclude

$$a_n = \frac{1}{r} \ge \frac{1}{2^{m+1}} \ge \frac{1}{2}\, n^{-\frac{1}{\alpha(n,d)}} \tag{4.19}$$

for all n, $C(1,d) < n \le C(2^d,d)$, hence, at least for $2 \le n \le \frac{d}{2}4^d$ (see (4.14)). If $s \neq 1$ we obtain (4.16) by raising (4.19) to the power s. This finishes the proof. ■

Remark 4.11. (i) Of course, there remains a gap between the lower bound in (4.16) and the upper bound in (4.12). For simplicity we comment on this gap for $s = 1$ only. In fact, we have

$$0 < \frac{1}{\alpha(n,d)} - \frac{1}{2 + \log_2 d} < \frac{\log_2 \log_2 n}{4 + 2\log_2 d}\,.$$

The gap is very mildly growing in n (keeping d fixed). Therefore, our estimates are loosing quality when n increases. Right now we do not have a conjecture about the correct bounds, most probably both, the lower and the upper estimate, can be improved.
(ii) Note that $\alpha(n,d)$ is decreasing in n. Hence, on certain smaller intervals of n, the dependence on n in $\alpha(n,d)$ can be removed by simple monotonicity arguments. For instance, since $\alpha(4^d,d) = 2$ and $\alpha(2^{2d/3},d) = 3$, we get

$$a_n(I_d : H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \ge 2^{-s} n^{-s/2}$$

simultaneously for all $2 \le n \le 4^d$, and

$$a_n(I_d : H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \ge 2^{-s} n^{-s/3}$$

simultaneously for all $2 \le n \le 2^{2d/3}$.
The approximation rate in these examples is much worse than the asymptotic rate n^{-s} (ignoring the logarithmic factors). This illustrates well that one has to wait exponentially long until one can "see" the correct asymptotic behavior of the approximation numbers.

4.3 The approximation numbers of $H^{s,+}_{\text{mix}}(\mathbb{T}^d)$, $H^{s,*}_{\text{mix}}(\mathbb{T}^d)$, and $H^{m}_{\text{mix}}(\mathbb{T}^d)$ in $L_2(\mathbb{T}^d)$

Now we turn to the investigation of $a_n(I_d : H^{s,+}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$ and $a_n(I_d : H^{s,*}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$.

4.3.1 Preparation

For $s > 0$ and $k \in \mathbb{Z}^d$ we put

$$w_s^+(k) := \prod_{j=1}^d (1+|k_j|^2)^{s/2} \qquad \text{and} \qquad w_s^*(k) := \prod_{j=1}^d (1+|k_j|^{2s})^{1/2}\,, \tag{4.20}$$

see (2.8) and (2.9), respectively. Of interest for us are the non-increasing rearrangements of $(1/w_s^+(k))_{k\in\mathbb{Z}^d}$ and $(1/w_s^*(k))_{k\in\mathbb{Z}^d}$. Let

$$C_s^+(r,d) := \#\{k \in \mathbb{Z}^d \;:\; w_{s,+}(k) \le r\} \qquad \text{and} \qquad C_s^*(r,d) := \#\{k \in \mathbb{Z}^d \;:\; w_{s,*}(k) \le r\}\,,$$

where $r \ge 1$, $s > 0$ are real numbers. Let us also define the smaller numbers

$$c_s^+(r,d) := \#\{k \in \mathbb{Z}^d \;:\; w_{s,+}(k) < r\} \qquad \text{and} \qquad c_s^*(r,d) := \#\{k \in \mathbb{Z}^d \;:\; w_{s,*}(k) < r\}\,.$$

In contrast to the weights $w_s^\#$, we now have no complete overview over all possible values of $w_s^+(k)$, $w_s^*(k)$ as k runs through $\mathbb{Z}^d$. Therefore, it is impossible to describe the full sequence of approximation numbers a_n. However, since $C_s^+((1+r^2)^{s/2},d) > c_s^+((1+r^2)^{s/2},d)$ and $C_s^*((1+r^{2s})^{1/2},d) > c_s^*((1+r^{2s})^{1/2},d)$ if $r \in \mathbb{N}_0$, we have at least some partial information about the piecewise constant sequence a_n of approximation numbers.

Lemma 4.12. *Let $s > 0$ and $r \in \mathbb{N}_0$.*
(i) *If $c_s^+((1+r^2)^{s/2},d) < n \le C_s^+((1+r^2)^{s/2},d)$, then*

$$a_n(I_d : H^{s,+}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = (1+r^2)^{-s/2}\,. \tag{4.21}$$

(ii) *If $c_s^+((1+r^{2s})^{1/2},d) < n \le C_s^+((1+r^{2s})^{1/2},d)$, then*

$$a_n(I_d : H^{s,*}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = (1+r^{2s})^{-1/2}\,.$$

Proof. By Lemma 2.6 and the same principles as used in (2.14), it is enough to note that we have $w_s^+(k) = (1+r^2)^{s/2}$ and $w_s^*(k) \ge (1+r^{2s})^{1/2}$ for $k = (r,0,\ldots,0) \in \mathbb{Z}^d$. ■

4.3.2 Some more combinatorics

Since we have only incomplete information on the set of all values attained by the weights $w_s^+(k)$ and $w_s^*(k)$, it is very difficult to establish similar combinatorial identities and sharp estimates as for the weight $w_s^\#(k)$. Therefore we decided for a different strategy. For $\ell \in \mathbb{Z}^d$, $0 < \varepsilon \le 1$ and $d \in \mathbb{N}$ let

$$a_\ell := \frac{1}{1+|\ell|} \quad , \quad \mathcal{A}_d(\varepsilon) := \Big\{k \in \mathbb{Z}^d :\, a_{k_1}\cdots a_{k_d} \ge \varepsilon\Big\} \quad , \quad A_d(\varepsilon) := \#\mathcal{A}_d(\varepsilon)\,.$$

Because of

$$\Big\{k \in \mathbb{Z}^d :\, a_{k_1}\cdots a_{k_d} \ge \frac{1}{r}\Big\} = \Big\{k \in \mathbb{Z}^d :\, \prod_{j=1}^d (1+|k_j|) \le r\Big\} = \mathcal{N}(r,d)$$

we have $A_d(1/r) = C(r, d)$ for all $r \in \mathbb{N}_0$. Using (4.1), Lemma 4.13 and a simple monotonicity argument, this implies

$$\lim_{\varepsilon \downarrow 0} \frac{\varepsilon \cdot A_d(\varepsilon)}{(\ln A_d(\varepsilon))^{d-1}} = \lim_{r \to \infty} \frac{C(r,d)}{r(\ln C(r,d))^{d-1}} = \frac{2^d}{(d-1)!} \,. \tag{4.22}$$

As consequences of these identities, we find for arbitrary $\lambda > 0$ and all $d \in \mathbb{N}$

$$\lim_{\varepsilon \downarrow 0} \frac{A_d(\varepsilon)}{A_d(\lambda\varepsilon)} = \lambda \qquad \text{and} \qquad \lim_{\varepsilon \downarrow 0} \frac{A_{d-1}(\varepsilon)}{A_d(\lambda\varepsilon)} = 0 \,. \tag{4.23}$$

Lemma 4.13. *Let $(b_\ell)_{\ell \in \mathbb{Z}}$ be a sequence indexed by $\mathbb{Z}$ such that*

$$0 < b_\ell \le b_0 = 1 \quad \textit{for all } \ell \neq 0 \qquad \textit{and} \qquad \lim_{|\ell| \to \infty} \frac{a_\ell}{b_\ell} = 1 \,.$$

Similarly as for $(a_\ell)_{\ell \in \mathbb{Z}^d}$ we define $\mathcal{B}_d(\varepsilon)$ and $B_d(\varepsilon)$ associated to $(b_\ell)_{\ell \in \mathbb{Z}^d}$. Then we have

$$\lim_{\varepsilon \downarrow 0} \frac{B_d(\varepsilon)}{A_d(\varepsilon)} = 1 \,. \tag{4.24}$$

Proof. Let us first observe that there are constants $0 < c \le C < \infty$ such that $c \le a_\ell / b_\ell \le C$ for all $\ell \in \mathbb{Z}$. Fix now $0 < \varepsilon \le 1$ and $\delta > 0$ (small), and select $m = m(\delta) \in \mathbb{N}$ such that

$$1 - \delta \le \frac{a_\ell}{b_\ell} \le 1 + \delta \qquad \text{for all } |\ell| \ge m \,.$$

For $k \in \mathcal{B}_d(\varepsilon)$, we distinguish two cases.

Case 1, $|k_j| \ge m$ for all j. This implies

$$\varepsilon \le \prod_{j=1}^{d} b_{k_j} \le \prod_{j=1}^{d} \frac{a_{k_j}}{1-\delta} \,,$$

and thus $k \in \mathcal{A}_d((1-\delta)^d \varepsilon)$.

Case 2, $|k_\ell| < m$ for some ℓ. Now we have

$$\varepsilon \le \prod_{j=1}^{d} b_{k_j} = b_{k_\ell} \prod_{j \neq \ell} b_{k_j} \le \prod_{j \neq \ell} \frac{a_{k_j}}{c} \,,$$

which gives $(k_1, ..., k_{\ell-1}, k_{\ell+1}, ..., k_d) \in \mathcal{A}_{d-1}(c^{d-1}\varepsilon)$. Since there are d choices of the index $\ell \in \{1, ..., d\}$ and $2m - 1$ possible values of k_ℓ, we conclude that

$$B_d(\varepsilon) \le A_d((1-\delta)^d \varepsilon) + (2m-1)d \cdot A_{d-1}(c^{d-1}\varepsilon) \,.$$

Using the relations (4.23), this gives

$$\limsup_{\varepsilon \downarrow 0} \frac{B_d(\varepsilon)}{A_d(\varepsilon)} \le \frac{1}{(1-\delta)^d} \,. \tag{4.25}$$

Now we show a lower estimate for $B_d(\varepsilon)$. Let $k \in \mathcal{A}_d((1+\delta)^d\varepsilon)$. Again we distinguish two cases.

If all $|k_j| \geq m$, we have

$$(1+\delta)^d\varepsilon \leq \prod_{j=1}^d a_{k_j} \leq \prod_{j=1}^d (1+\delta) b_{k_j} \,,$$

that means $k \in \mathcal{B}_d(\varepsilon)$.

Otherwise, if $k_\ell < m$ for some ℓ, we have

$$(1+\delta)^d\varepsilon \leq a_{k_\ell} \cdot \prod_{j\neq\ell} a_{k_j} \leq \prod_{j\neq\ell} a_{k_j} \,,$$

which means $(k_1, ..., k_{\ell-1}, k_{\ell+1}, ..., k_d) \in \mathcal{A}_{d-1}((1+\delta)^d\varepsilon)$, and we get

$$A_d((1+\delta)^d\varepsilon) - 2(m-1)d \cdot A_{d-1}((1+\delta)^d\varepsilon) \leq B_d(\varepsilon) \,.$$

This implies, using again (4.23),

$$\liminf_{\varepsilon\downarrow 0} \frac{B_d(\varepsilon)}{A_d(\varepsilon)} \geq \frac{1}{(1+\delta)^d} \,, \tag{4.26}$$

and since (4.25) and (4.26) are true for all $\delta > 0$, the proof is finished. ■

There are some simple consequences of Lemma 4.13 which are of interest for us. Taking logarithms in (4.24) yields

$$\lim_{\varepsilon\downarrow 0} \Big(\ln B_d(\varepsilon) - \ln A_d(\varepsilon) \Big) = 0.$$

Since $\lim_{\varepsilon\downarrow 0} A_d(\varepsilon) = \infty$, we get

$$\lim_{\varepsilon\downarrow 0} \frac{\ln B_d(\varepsilon)}{\ln A_d(\varepsilon)} = \lim_{\varepsilon\downarrow 0} \frac{\ln B_d(\varepsilon) - \ln A_d(\varepsilon)}{\ln A_d(\varepsilon)} + 1 = 1 \,. \tag{4.27}$$

Hence,

$$\frac{\varepsilon \cdot B_d(\varepsilon)}{(\ln B_d(\varepsilon))^{d-1}} = \frac{\varepsilon \cdot A_d(\varepsilon)}{(\ln A_d(\varepsilon))^{d-1}} \cdot \left(\frac{\ln A_d(\varepsilon)}{\ln B_d(\varepsilon)}\right)^{d-1} \cdot \frac{B_d(\varepsilon)}{A_d(\varepsilon)} \,.$$

Together with (4.22), (4.24), and (4.27) this implies

$$\lim_{\varepsilon\downarrow 0} \frac{\varepsilon \cdot B_d(\varepsilon)}{(\ln B_d(\varepsilon))^{d-1}} = \lim_{\varepsilon\downarrow 0} \frac{\varepsilon \cdot A_d(\varepsilon)}{(\ln A_d(\varepsilon))^{d-1}} = \frac{2^d}{(d-1)!} \,. \tag{4.28}$$

4.3.3 The approximation numbers of $H^{s,+}_{\mathrm{mix}}(\mathbb{T}^d)$ in $L_2(\mathbb{T}^d)$

In our first application of (4.22) we choose $b_\ell := (1+|\ell|^2)^{-1/2}$, $\ell \in \mathbb{Z}$. Then

$$\mathcal{B}_d(\varepsilon) := \Big\{ k \in \mathbb{Z}^d : \prod_{j=1}^d b_{k_j} \geq \varepsilon \Big\} = \Big\{ k \in \mathbb{Z}^d : 1/w_1^+(k) \geq \varepsilon \Big\} \,,$$

for all $\varepsilon > 0$.

Corollary 4.14. *Let $d \in \mathbb{N}$.*
(i) *Let $s > 0$. Then*

$$\lim_{n\to\infty} \frac{n^s\, a_n(I_d : H^{s,+}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))}{(\ln n)^{(d-1)s}} = \Big[\frac{2^d}{(d-1)!}\Big]^s .$$

(ii) *Let $s_0 > s_1 \geq 0$. Then*

$$\lim_{n\to\infty} \frac{n^{s_0-s_1}\, a_n(I_d : H^{s_0,+}_{\text{mix}}(\mathbb{T}^d) \to H^{s_1,+}_{\text{mix}}(\mathbb{T}^d))}{(\ln n)^{(d-1)(s_0-s_1)}} = \Big[\frac{2^d}{(d-1)!}\Big]^{s_0-s_1} .$$

Proof. It is enough to prove (i) for $s = 1$. Indeed, then the known relation

$$a_n(I_d : H^{s,+}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = a_n(I_d : H^{1,+}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))^s .$$

implies (i) for arbitrary $s > 0$, and the semigroup property of the weights yields (ii).
Setting $\varepsilon_r := (1+r^2)^{-1/2}$ for $r \in \mathbb{N}_0$, we obviously have

$$\{\varepsilon_r : r \in \mathbb{N}\} \subset \{1/w_1^+(k) : k \in \mathbb{Z}^d\} ,$$

whence $a_n := a_n(I_d : H^{1,+}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \varepsilon_r$ if $n = B_d(\varepsilon_r)$. Consequently, if $B_d(\varepsilon_{r-1}) \leq n \leq B_d(\varepsilon_r)$, then

$$\varepsilon_r \leq a_n \leq \varepsilon_{r-1} \qquad \text{and} \qquad \frac{\varepsilon_r \cdot B_d(\varepsilon_{r-1})}{(\ln B_d(\varepsilon_{r-1}))^{d-1}} \leq \frac{n a_n}{(\ln n)^{d-1}} \leq \frac{\varepsilon_{r-1} \cdot B_d(\varepsilon_r)}{(\ln B_d(\varepsilon_r))^{d-1}} .$$

Since $\lim_{r\to\infty} \varepsilon_{r-1}/\varepsilon_r = 1$, a simple monotonicity argument and (4.28) imply

$$\lim_{n\to\infty} \frac{n a_n}{(\ln n)^{d-1}} = \lim_{\varepsilon\downarrow 0} \frac{\varepsilon \cdot B_d(\varepsilon)}{(\ln B_d(\varepsilon))^{d-1}} = \lim_{\varepsilon\downarrow 0} \frac{\varepsilon \cdot A_d(\varepsilon)}{(\ln A_d(\varepsilon))^{d-1}} = \frac{2^d}{(d-1)!} .$$

■

Corollary 4.14 is the basis for the two-sided estimates of $a_n(I_d : H^{s,+}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$ for large n which we will study next.

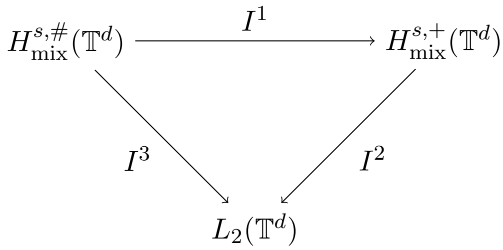
Theorem 4.15. *Let $s > 0$ and $d \in \mathbb{N}$. Then we have*

$$a_n(I_d : H^{s,+}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq \Big[\frac{(3\cdot\sqrt{2})^d}{(d-1)!}\Big]^s \frac{(\ln n)^{(d-1)s}}{n^s} , \qquad \textit{if} \quad n \geq 27^d$$

and

$$a_n(I_d : H^{s,+}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \geq \left[\frac{3}{d!}\left(\frac{2}{2+\ln 12}\right)^d\right]^s \frac{(\ln n)^{(d-1)s}}{n^s} , \qquad \textit{if} \quad n > (12\, e^2)^d .$$

Proof. By I^j, $j = 1,2,3$, we denote identity mappings.
Since we have $\| I^1 : H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \to H^{s,+}_{\text{mix}}(\mathbb{T}^d)\| = 1$, the commutative diagram



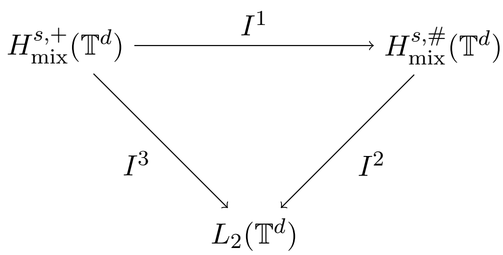
with $I^3 = I^2 \circ I^1$ and basic properties of approximation numbers yields

$$a_n(I^3) \le \| \, I^1 \, | H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \to H^{s,+}_{\text{mix}}(\mathbb{T}^d) \| \, a_n(I^2)\,.$$

Supplemented by Theorem 4.5/(ii) the lower estimate of $a_n(I_d : H^{s,+}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$ follows. What concerns the upper bound we observe

$$\| \, I^1 : H^{s,+}_{\text{mix}}(\mathbb{T}^d) \to H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \| = 2^{ds/2}$$

and consider the diagram



with $I^3 = I^2 \circ I^1$. This leads to

$$a_n(I^3) \le 2^{ds/2} \; a_n(I^2)\,.$$

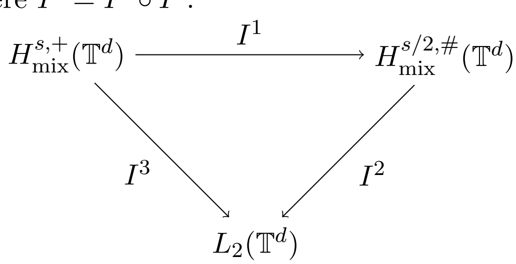
■

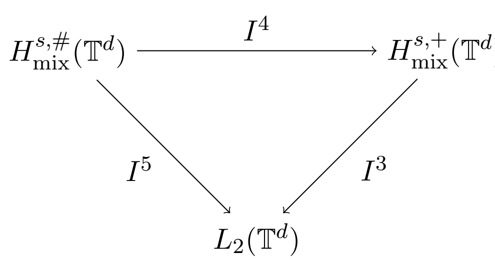
Finally, we shall have a look at the behavior of $a_n(I_d : H^{s,+}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$ for small n. Recall that the quantity $\alpha(n,d)$ has been defined in (4.15).

Theorem 4.16. *Let $d \in \mathbb{N}$, $d \ge 2$, and $s > 0$. Then for any $2 \le n \le \frac{d}{2} 4^d$ it holds the two-sided estimate*

$$2^{-s}\Big(\frac{1}{n}\Big)^{\frac{s}{2+\alpha(n,d)}} \le a_n(I_d : H^{s,+}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le \Big(\frac{e^2}{n}\Big)^{\frac{s}{4+2\log_2 d}}\,.$$

Proof. The upper bound is a direct consequence of basic properties of approximation numbers, see Subsection 2.3, Lemma 2.2/(v), Theorem 4.9 and the first commutative diagram below, where $I^3 = I^2 \circ I^1$.





The lower bound follows from Lemma 2.2/(i)-(iii), Theorem 4.10 and the second commutative diagram above, where $I^5 = I^3 \circ I^4$. ■

4.3.4 The approximation numbers of $H^{s,*}_{\text{mix}}(\mathbb{T}^d)$ in $L_2(\mathbb{T}^d)$

In our second application of (4.22) we choose $b_\ell := (1+|\ell|^{2s})^{-1/(2s)}$. This leads to

$$\mathcal{B}_d(\varepsilon) := \Big\{k \in \mathbb{Z}^d : \prod_{j=1}^d b_{k_j} \ge \varepsilon\Big\} = \Big\{k \in \mathbb{Z}^d : 1/w^*_s(k) \ge \varepsilon^s\Big\}.$$

Due to the missing semigroup property we have to deal now with all $s > 0$, not only with $s = 1$. But nevertheless we can proceed similarly as in the previous subsection.

Corollary 4.17. *Let $d \in \mathbb{N}$ and $s > 0$. Then*

$$\lim_{n\to\infty} \frac{n^s\, a_n(I_d : H^{s,*}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))}{(\ln n)^{(d-1)s}} = \Big[\frac{2^d}{(d-1)!}\Big]^s.$$

Proof. Again (4.28) leads to

$$\lim_{\varepsilon\downarrow 0} \frac{\varepsilon B_d(\varepsilon)}{(\ln B_d(\varepsilon))^{d-1}} = \frac{2^d}{(d-1)!}.$$

Setting $\varepsilon_r := (1+r^{2s})^{-1/2s}$, we have $\lim_{r\to\infty} \frac{\varepsilon_r}{\varepsilon_{r-1}}$, and

$$\{\varepsilon_r^s : r \in \mathbb{N}_0\} \subset \{1/w^*_s(k) : k \in \mathbb{Z}^d\}.$$

Therefore, $\varepsilon_r^s \le a_n \le \varepsilon_{r-1}$, if $B_d(\varepsilon_r) \le n \le B_d(\varepsilon_{r-1})$. This gives

$$\varepsilon_r \le a_n^{1/s} \le \varepsilon_{r-1} \qquad \text{and} \qquad \frac{\varepsilon_r \cdot B_d(\varepsilon_{r-1})}{(\ln B_d(\varepsilon_{r-1}))^{d-1}} \le \frac{n a_n^{1/s}}{(\ln n)^{d-1}} \le \frac{\varepsilon_{r-1}\cdot B_d(\varepsilon_r)}{(\ln B_d(\varepsilon_r))^{d-1}}.$$

Exactly the same arguments as in the proof of Theorem 4.14 imply

$$\lim_{n\to\infty} \frac{n a_n^{1/s}}{(\ln n)^{d-1}} = \lim_{\varepsilon\downarrow 0} \frac{\varepsilon \cdot B_d(\varepsilon)}{(\ln B_d(\varepsilon))^{d-1}} = \frac{2^d}{(d-1)!},$$

and this is equivalent to our assertion. ■

.

Based on Corollary 4.17 we can derive two-sided estimates of $a_n(I_d : H^{s,*}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$ for large n.

Theorem 4.18. *Let $d \in \mathbb{N}$.*
(i) Let $s > 1/2$. Then

$$a_n(I_d : H^{s,*}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le 2^{-d/2}\Big[\frac{6^d}{(d-1)!}\Big]^s \frac{(\ln n)^{(d-1)s}}{n^s} \qquad \text{if} \quad n \ge 27^d,$$

and

$$a_n(I_d : H^{s,*}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \ge \Big[\frac{3\cdot 2^d}{d!\,(2+\ln 12)^d}\Big]^s \frac{(\ln n)^{(d-1)s}}{n^s} \qquad \text{if} \quad n > (12\,e^2)^d.$$

(ii) Let $0 < s \le 1/2$. *Then*

$$a_n(I_d:\, H^{s,*}_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le \Big[\frac{3^d}{(d-1)!}\Big]^s \,\frac{(\ln n)^{(d-1)s}}{n^s} \qquad \textit{if} \quad n \ge 27^d\,,$$

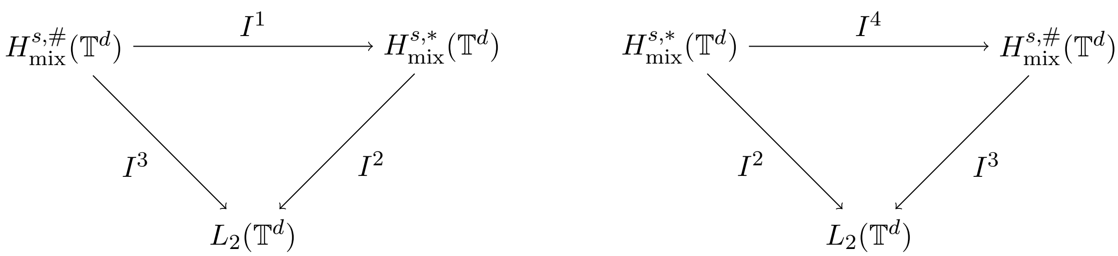
and

$$a_n(I_d:\, H^{s,*}_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \ge 2^{-d/2}\Big[\frac{3\cdot 4^d}{d!\,(2+\ln 12)^d}\Big]^s \,\frac{(\ln n)^{(d-1)s}}{n^s} \qquad \textit{if} \quad n > (12\,e^2)^d\,.$$

Proof. We distinguish two cases: $s > 1/2$ and $0 < s \le 1/2$.
Step 1. Let $s > 1/2$. Then

$$\|I^1: H^{s,\#}_{\mathrm{mix}}(\mathbb{T}^d) \to H^{s,*}_{\mathrm{mix}}(\mathbb{T}^d)\| = 1 \qquad \text{and} \qquad \|I^4: H^{s,*}_{\mathrm{mix}}(\mathbb{T}^d) \to H^{s,\#}_{\mathrm{mix}}(\mathbb{T}^d)\| = 2^{(s-1/2)d}\,.$$

In view of the diagrams



with $I^3 = I^2 \circ I^1$, $I^2 = I^3 \circ I^4$, this yields

$$a_n(I^3) \le\, a_n(I^2) \le 2^{(s-1/2)d}\, a_n(I^3)\,.$$

Now the claimed estimates follow from Theorem 4.5.
Step 2. Let $0 < s \le 1/2$. Then

$$\|I^1: H^{s,\#}_{\mathrm{mix}}(\mathbb{T}^d) \to H^{s,*}_{\mathrm{mix}}(\mathbb{T}^d)\| = 2^{(\frac{1}{2}-s)d} \qquad \text{and} \qquad \|I^4: H^{s,*}_{\mathrm{mix}}(\mathbb{T}^d) \to H^{s,\#}_{\mathrm{mix}}(\mathbb{T}^d)\| = 1\,.$$

Employing the same diagrams as in Step 1 we conclude

$$a_n(I^2) \le\, a_n(I^3) \le 2^{(\frac{1}{2}-s)d}\, a_n(I^2)\,.$$

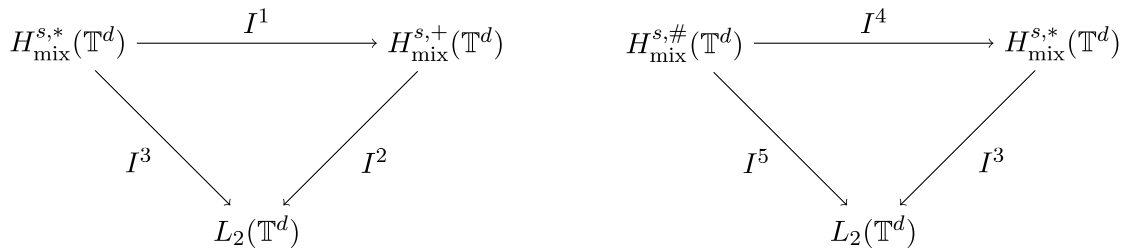
Now the claimed estimates follow from Theorem 4.5. ■

Again, as the last step in this subsection, we shall consider the behavior of $a_n(I_d:\, H^{s,*}_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$ for small n. This time we have only partial results.

Theorem 4.19. *Let* $d \in \mathbb{N}$, $d \ge 2$, *and* $1/2 \le s \le 1$. *Then for any* $2 \le n \le \frac{d}{2}4^d$ *it holds the two-sided estimate*

$$2^{-s}\Big(\frac{1}{n}\Big)^{\frac{s}{2+\alpha(n,d)}} \le a_n(I_d: H^{s,*}_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le \Big(\frac{e^2}{n}\Big)^{\frac{s}{4+\log_2(d^2)}}\,.$$

Proof. We argue as in the proof of Theorem 4.16. The upper bound is a consequence of Lemma 2.2,(ii) together with Theorem 4.16 and the first commutative diagram



with $I^3 = I^2 \circ I^1$. The lower bound follows from Lemma 2.2, (ii), Theorem 4.10 and the second commutative diagram using $I^5 = I^3 \circ I^4$. ■

4.4 The approximation numbers of $H^m_{\mathrm{mix}}(\mathbb{T}^d)$ in $L_2(\mathbb{T}^d)$

By setting $b_\ell = v_m(\ell)^{1/m}$, see (4.21), we could argue similar as in the previous subsection to compute $\lim_{n\to\infty} n^m \cdot a_n/(\ln n)^{(d-1)m}$. However, Lemma 2.3 provides a much simpler argument.

Corollary 4.20. *Let $d \in \mathbb{N}$ and $m \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \frac{n^m\, a_n(I_d : H^m_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))}{(\ln n)^{(d-1)m}} = \Big[\frac{2^d}{(d-1)!}\Big]^m .$$

Proof. This follows immediately from Corollaries 4.14 and 4.17, and

$$\begin{aligned} a_n(I_d : H^{m,+}_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) &\le a_n(I_d : H^m_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \\ &\le a_n(I_d : H^{m,*}_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \end{aligned} \tag{4.29}$$

which is itself a consequence of Lemma 2.3. ■

Based on Thms. 4.15, 4.18 and (4.29) we derive two-sided estimates of $a_n(I_d : H^m_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$ for large n.

Theorem 4.21. *Let $m \in \mathbb{N}$ and $d \in \mathbb{N}$.*
(i) Then

$$a_n(I_d : H^m_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le \Big[\frac{6^d}{(d-1)!}\Big]^m 2^{-d/2}\, \frac{(\ln n)^{(d-1)m}}{n^m} \qquad \text{if} \quad n \ge 27^d .$$

(ii) In addition

$$a_n(I_d : H^m_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \ge \Big[\frac{3 \cdot 2^d}{d!\,(2+\ln 12)^d}\Big]^s \frac{(\ln n)^{(d-1)m}}{n^m} \qquad \text{if} \quad n > (12\, e^2)^d .$$

Remark 4.22. Also for the embedding $H^m_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$ the behavior of $a_n(I_d : H^m_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$ for small n is of interest. By the coincidence $H^1_{\mathrm{mix}}(\mathbb{T}^d) = H^{1,*}_{\mathrm{mix}}(\mathbb{T}^d)$ (equal norms) we obtain the relations

$$\frac{1}{2}\Big(\frac{1}{n}\Big)^{\frac{1}{2+\alpha(n,d)}} \le a_n(I_d : H^1_{\mathrm{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le \Big(\frac{e^2}{n}\Big)^{\frac{1}{4+\log_2(d^2)}}$$

immediately from Theorem 4.19.

4.5 Various comments on the literature

- Closest to us in aims and methods is the recent paper [7]. There, in Theorem 3.13, the authors obtained for $s > 0$ and any $n \geq 2^d$ the inequality

$$a_n(I_d : H^{s,\square}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \ \leq \ 4^s \Big(\frac{2e}{d-1}\Big)^{s(d-1)} n^{-s} (\log_2 n)^{s(d-1)}, \tag{4.30}$$

  where $\square$ indicates that the space $H^{s,\square}_{\text{mix}}(\mathbb{T}^d)$ is equipped with a further norm (based on dyadic decompositions on the Fourier side and different from those studied here). This has to be compared with Theorems 4.5, 4.15, 4.18 and 4.21. In all these cases we have a super-exponential decay of the constants $C_s(d)$ in d.

- Super-exponential decay of the constants $C_s(d)$ in d has been observed before. Bungartz and Griebel [4, Theorem 3.8] investigated the non-periodic situation. An approximation is given with respect to tensor products of piecewise linear functions. The authors proved that for any $n \in \mathbb{N}$ there exists a subspace $V_n^{(1)} \subset L_2([0,1]^d)$ with $m = m(d,n)$ degrees of freedom and a projection Q_n onto $V_n^{(1)}$ such that

$$\| \, f - Q_n f \, | L_2([0,1]^d) \| \leq \frac{2}{12^d} 2^{-2n} A(d,n) \, \Big\| \, \frac{\partial^{2d} f}{\partial x_1^2 \, \ldots \, \partial x_d^2} \, \Big| L_2([0,1]^d) \Big\| \tag{4.31}$$

  holds for all continuous functions f vanishing on the boundary $\partial([0,1]^d)$. The latter assumption is actually crucial for the explicit bound in (4.31). Here, the number $A(d,n)$ is given by

$$A(d,n) := \sum_{k=0}^{d-1} \binom{n+d-1}{k} .$$

  Of course, inequality (4.31) does not allow for a comparison with the results obtained in this paper. One first has to rewrite the bounds in terms of the degrees of freedom m. For this issue a careful two-sided estimate of $m = \dim V_n^{(1)}$ is required first. Lemma 3.6 in [4] shows that $\dim V_n^{(1)}$ can be estimated from above and below by

$$2^{n-1} \binom{n+d-2}{d-1} \leq \dim V_n^{(1)} = \sum_{j=0}^{n-1} 2^j \binom{j+d-1}{d-1} \leq 2^n \binom{n+d-2}{d-1} . \tag{4.32}$$

  For $n \geq d-1$ the expression $\binom{n+d-1}{k}$ is increasing in $k \leq d-1$. In this case we have the estimate

$$A(d,n) \leq d \binom{n+d-1}{d-1} .$$

  We will now transfer (4.31) to the notion of approximation numbers. To be precise we consider the space/norm

$$\begin{aligned} H^2_{\text{mix},0}([0,1]^d) := \{ f \in L_2([0,1]^d) \ : \ \| \, f \, | H^2_{\text{mix}}([0,1]^d) \| := \max_{\alpha \in \{0,2\}^d} \| \, D^\alpha f \, | L_2([0,1]^d) \| < \infty \\ \text{and } f = 0 \quad \text{on the boundary} \} . \end{aligned}$$

Note, that $\| f \,| H^2_{\text{mix}}([0,1]^d)\|$ in the previous formula represents a very weak norm in $H^2_{\text{mix},0}([0,1]^d)$ compared to (2.2). Based on (4.31), (4.32) and the monotonicity of approximation numbers we find for any m satisfying $\dim V_n^{(1)} < m \le \dim V_{n+1}^{(1)}$ for some $n \ge d-1$ the relation

$$\begin{aligned} a_m(I_d : H^2_{\text{mix},0}([0,1]^d) \to L_2([0,1]^d)) &\le \frac{2}{12^d} 2^{-2n} A(d,n) \le \frac{8d}{12^d}\, m^{-2} \binom{n+d-1}{d-1}^3 \\ &\le \frac{8d}{12^d}\, m^{-2} \binom{\log_2 m + d}{d-1}^3 \\ &\le \frac{8d}{12^d}\, m^{-2} \Big(\frac{2\, e\, \log_2 m}{d-1}\Big)^{3(d-1)} . \end{aligned}$$

Consequently, inequality (4.31) implies for any $m \ge 2^{d-1}\binom{2d-2}{d-1}$ that

$$a_m(I_d : H^2_{\text{mix},0}([0,1]^d) \to L_2([0,1]^d)) \le C(d)\, m^{-2} (\log_2 m)^{3(d-1)} , \tag{4.33}$$

where

$$C(d) := \frac{2d}{3} \Big(\frac{2e}{12^{1/3}(d-1)}\Big)^{3(d-1)} .$$

This constant $C(d)$ is decaying extremely fast, i.e., super-exponentially in d, similar as in Thms. 4.5, 4.15, 4.18, 4.21 or in (4.30) above. But comparing (4.33) with the quoted estimates of the approximation numbers $a_n(I_d : H^2_{\text{mix}}(\mathbb{T}^d), L_2(\mathbb{T}^d))$, then it is obvious that the power of the logarithm $3(d-1)$ in (4.33) is larger than there, where it is always $2(d-1)$. This is at least partly caused by the fact that interpolation operators of Smolyak type are known to be not optimal in the sense of approximation numbers in such a context and Bungartz and Griebel are using an interpolation operator of Smolyak type with respect to a sparse grid. However, a reasonable comparison of (4.31) and Thms. 4.5, 4.15, 4.18, 4.21 or (4.30) can not be made because of the following reasons:

(i) The periodic Sobolev spaces $H^2_{\text{mix}}(\mathbb{T}^d)$ are smaller than the non-periodic Sobolev spaces $H^2_{\text{mix}}([0,1]^d)$ (and the "difference" is increasing with d).

(ii) The space $H^2_{\text{mix},0}([0,1]^d)$ is much smaller than the original space $H^2_{\text{mix}}([0,1]^d)$.

(iii) On the right-hand side in (4.31) only the term $\| \dfrac{\partial^{2d} f}{\partial x_1^2 \ldots \partial x_d^2} \,| L_2([0,1]^d)\|$ shows up, which is much smaller than the full norm used in our investigations above.

- Preasymptotics. The inequalities (4.31) and (4.32) remain also true for small n. Note, that in case $1 \le n \le d-1$ the number $A(d,n)$ can be estimated as follows

$$\frac{1}{2}\, 2^{n+d-1} \le A(d,n) = \sum_{k=0}^{d-1} \binom{n+d-1}{k} \le 2^{n+d-1}$$

(we sum up to $d-1$, which is larger than $(n+d-1)/2$). Let $\dim V_n^{(1)} \le m \le \dim V_{n+1}^{(1)}$ for some $1 \le n \le d-1$. Using (4.32), (4.31) and the space $H^2_{\text{mix},0}([0,1]^d)$ defined above we obtain

$$
\begin{aligned}
a_m(I_d : H^2_{\text{mix},0}([0,1]^d) \to L_2([0,1]^d)) &\le \frac{2}{12^d} 2^{-2n} A(d,n) \\
&\le \frac{4}{12^d} m^{-1} 2^{n+d-1} 2^{-n} \binom{n+d-1}{d-1} \\
&\le \frac{1}{3} \Big(\frac{4e}{12}\Big)^{d-1} m^{-1} = \frac{1}{e} \Big(\frac{e}{3}\Big)^d m^{-1} .
\end{aligned}
$$

Hence we obtain for $1 \le m \le 2^{d-1}\binom{2d-2}{d-1}$ the "preasymptotic" decay

$$
a_m(I_d : H^2_{\text{mix},0}([0,1]^d) \to L_2([0,1]^d)) \le C(d) m^{-1} \tag{4.34}
$$

with

$$
C(d) := \frac{1}{e} \Big(\frac{e}{3}\Big)^d .
$$

This time we "just have" exponential decay of the constant $C(d)$ in d. Now we compare this with our results obtained in Thms. 4.9, 4.16 and 4.19. Let us concentrate on (4.12). There we proved for a range in $1 \le m \le 4^d$ the inequality

$$
a_m(I_d : H^{2,\#}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le \Big(\frac{e^2}{m}\Big)^{\frac{1}{1+\log_2 \sqrt{d}}} .
$$

Inequality (4.34) looks much better than this inequality (with respect to the exponent of m as well as with respect to the d-dependence of the constant). This is mainly caused by the homogeneous boundary condition in (4.34) which shrinks the space significantly.

- Also Schwab, Süli, and Todor [23] considered the non-periodic situation. A particular case of their results in [23, Theorem 5.1] can be formulated as follows. For $s \in \mathbb{N}$, $s \ge 2$ and $L \ge d-1$ there exists a subspace V_L of $L_2([0,1]^d)$ with $m = \dim V_L$ degrees of freedom and a projection Q_L onto V_L such that

$$
\| f - Q_L f \,| L_2([0,1]^d) \| \le C_2(d,s) m^{-s} (\log m)^{(d-1)(s+1)} \| f \,| H^{s,\triangle}_{\text{mix}}([0,1]^d) \| , \tag{4.35}
$$

where $C_2(d,s)$ decays super-exponentially in d. Here the authors use the stronger norm

$$
\| f \,| H^{s,\triangle}_{\text{mix}}([0,1]^d) \| := \sum_{\substack{0 \le \alpha_i \le s \\ i=1,\ldots,d}} \| D^\alpha f \,| L_2([0,1]^d) \| ,
$$

where $s \in \mathbb{N}$. The result is stated in [23] in a slightly different form. As above one has to supplement an inequality similar to (4.31) by two-sided estimates for $\dim V_L$ to turn it into (4.35). In some sense (4.35) generalizes (4.31) to the case of higher smoothness. As before the power of the logarithm is worse compared with the behavior of the approximation numbers $a_m(I_d : H^s_{\text{mix}}(\mathbb{T}^d), L_2(\mathbb{T}^d))$.

- Preasymptotics. In case $L \le d-1$ Schwab, Süli, and Todor [23] obtained, under additional restrictions, the estimate

$$
\| f - Q_L f \,| L_2([0,1]^d) \| \le C_3(d,s) 2^{-Ls} \| f \,| H^{s,\triangle}_{\text{mix}}([0,1]^d) \| ,
$$

where $C_3(d,s)$ decays exponentially in d. Again this has to be complemented by a two-sided estimate of $m := \dim V_L$. A rather rough but sophisticated estimate yields

$$\| f - Q_L f \,|L_2([0,1]^d)\| \le C_4(d,s) m^{-s/3} \, \| f \,|H^{s,\triangle}_{\text{mix}}([0,1]^d)\| \,, \qquad m \le 2^{2(d-1)} \,,$$

where we have been unable to clarify the dependence of the constant $C_4(d,s)$ on d.

- Neither Bungartz and Griebel [4] nor Schwab, Süli, and Todor [23] considered estimates from below.

- Sampling operators versus general linear operators. As mentioned above the estimates (4.33) and (4.35) are obtained by using interpolation operators Q_n with $m(d,n)$ sample points based upon univariate spline interpolation operators (via a Smolyak construction). Let us mention the following result in this context: in [27, Theorem 6.2,(i)] we show for $1/2 < s < 2$ and all $m \in \mathbb{N}$

$$\| f - A_m f \,|L_2([0,1]^d)\| \lesssim m^{-s} (\log m)^{(d-1)(s+1/2)} \, \| f \,|H^{s}_{\text{mix}}([0,1]^d)\| \,. \tag{4.36}$$

Compared to the estimates (4.33) and (4.35) we improved the power of the logarithm by $(d-1)/2$, however, we do not know about the d-dependence of the constants in (4.36). The restrictions on s in (4.36) are caused by the fact that we worked with piecewise linear functions. In Dũng [5] the relation (4.36) has been extended to all $s > 1/2$ via B-spline quasi-interpolation (but also without taking care of the d-dependence of the constants).

- Motivated by the aim to approximate the solution of a Poisson equation in the energy norm, i.e., in the norm of the isotropic Sobolev space H^1, Bungartz and Griebel [3] investigated estimates of the quantities $a_n(I_d : H^2_{\text{mix}}([0,1]^d), H^1([0,1]^d))$. These studies have been continued in Griebel, Knapek [10, 11], Bungartz, Griebel [4], Griebel [9], Schwab, Süli, and Todor [23], and Dũng, Ullrich [7]. Let us comment on the non-periodic situation first. It was already noticed by Griebel in [9, Theorem 2] that in this situation the constant (in front of the approximation order term) decays exponentially in d. To be more precise, he proved that there is a subspace V_n with n degrees of freedom and a projection Q_n onto V_n such that for large n

$$\| f - Q_n f \,|H^1([0,1]^d)\| \le c \cdot c_1(d) \cdot c_2(d) \cdot n^{-1} \left\| \frac{\partial^{2d} f}{\partial x_1^2 \ldots \partial x_d^2} \,\Big| L_\infty[0,1]^d \right\| , \tag{4.37}$$

holds, where

$$c_1(d) = \frac{d}{2} e^d \quad \text{and} \quad c_2(d) = \frac{d}{3^{(d-1)/2}\, 4^{d-1}} \Big[\frac{1}{2} + \Big(\frac{5}{2}\Big)^{d-1} \Big] \,.$$

Hence, the product $c_1(d) c_2(d)$ decays like $d^2 \cdot 0.980875^d$. Note, that the L_∞-norm is involved in (4.37) and the functions f are taken from spaces with mixed smoothness of order 2 and homogeneous boundary conditions. The situation changes significantly if one replaces L_∞ by L_2 in (4.37). The source space for f is now getting larger and hence the approximation is getting worse. In [4, Table 3.2, page 35] the constant $c_2(d)$ can be chosen as

$$c_2(d) = \frac{2d}{\sqrt{3} \cdot 6^{d-1}} \Big[\frac{1}{2} + \Big(\frac{5}{2}\Big)^{d-1} \Big] \,.$$

Therefore, $c_1(d)c_2(d)$ can only be estimated by $d^2 \cdot 1.1326^d$ and thus an exponential decay can not longer be guaranteed. However, if the smoothness s of the source space is less than 2 we can say something in the periodic setting. From [7, Theorem 3.6,(ii)] it follows that if the error is measured in $H^1(\mathbb{T}^d)$ and $s < 2$ we get for $n > \lambda^d$ (for some $\lambda > 1$) the relation

$$a_n(I_d : \tilde{H}^s_{\text{mix}}(\mathbb{T}^d) \to H^1(\mathbb{T}^d)) \leq c d^{s-1}\Big(\frac{1}{2^{1/(s-1)} - 1}\Big)^d n^{-(s-1)} .$$

Here $\tilde{H}^s_{\text{mix}}(\mathbb{T}^d)$ is the subspace of $H^s_{\text{mix}}(\mathbb{T}^d)$ containing all functions f such that $c_k(f) \neq 0 \implies \prod_{i=1}^d k_i \neq 0$. This technical condition is essential to prove (4.37), see [9, Theorem 2]. Without this condition, i.e., for the entire space $H^s_{\text{mix}}(\mathbb{T}^d)$ normed with dyadic building blocks on the Fourier side, we can disprove the exponential decay of the constants if $s \leq 2$, see [7, Theorem 4.7,(i)].

5 Quasi-polynomial tractability

Now we will translate our results to recent tractability notions. Various concepts of tractability are discussed in the recent monographs by Novak and Woźniakowski [17, 19, 20]. We will obtain "quasi-polynomial tractability" of the respective approximation problems. This notion has been recently introduced in [8] and is a stronger notion than "weak tractability".

5.1 General notions of tractability

For arbitrary $s > 0$ and all $d \in \mathbb{N}$ we consider the embedding operators (formal identities)

$$I_d : H^s_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d) ,$$

where the Sobolev spaces are equipped with the norms $\| \cdot |H^s_{\text{mix}}(\mathbb{T}^d)\|$, $\| \cdot |H^s_{\text{mix}}(\mathbb{T}^d)\|^*$, and $\| \cdot |H^s(\mathbb{T}^d)\|^\#$. In both cases we have $\|I_d\| = 1$ for all $s > 0$ and $d \in \mathbb{N}$. In other words, the *normalized error criterion* is satisfied. In this context, a *linear algorithm* that uses *arbitrary information* is of the form

$$A_{n,d}(f) = \sum_{j=1}^n L_j(f) g_j ,$$

where $g_j \in L_2(\mathbb{T}^d)$ and L_j are continuous linear functionals. If the error is measured in the norm of $L_2(\mathbb{T}^d)$ we can identify the algorithm $A_{n,d}$ with a bounded linear operator $A_{n,d} : H^s_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$ of rank at most n. The worst-case error of $A_{n,d}$ with respect to the unit ball (respective norms) in $H^s_{\text{mix}}(\mathbb{T}^d)$

$$\sup_{\|f|H^s_{\text{mix}}(\mathbb{T}^d)\| \leq 1} \|f - A_{n,d}(f)|L_2(\mathbb{T}^d)\|$$

clearly coincides with the operator norm $\|I_d - A_{n,d} : H^s_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\|$, and the *nth minimal worst-case error* with respect to *linear algorithms and general information*

$$\inf_{\text{rank}\, A_{n,d} \leq n} \|I - A_{n,d} : H^s_{\text{mix}}(\mathbb{T}^d) \to L(\mathbb{T}^d)\|$$

is just the *approximation number* $a_{n+1}(I_d : H^s_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$, see (1.1).

Finally, the *information complexity* of the d-variate approximation problem is measured by the quantity $n(\varepsilon, d)$ defined by

$$n(\varepsilon, d) := \inf\{n \in \mathbb{N} : a_n(I_d) \le \varepsilon\}$$

as $\varepsilon \to 0$ and $d \to \infty$. The approximation problem is called *weakly tractable*, if

$$\lim_{1/\varepsilon + d \to \infty} \frac{\ln n(\varepsilon, d)}{1/\varepsilon + d} = 0\,, \tag{5.1}$$

i.e., $n(\varepsilon, d)$ neither depends exponentially on $1/\varepsilon$ nor on d. The problem is called *intractable*, if (5.1) does not hold, see the definition [17, p. 7]. If for some $0 < \varepsilon < 1$ the number $n(\varepsilon, d)$ is an exponential function in d then we say that the approximation problem suffers from *the curse of dimensionality.* In other words, if there exist positive numbers c, ε_0, γ such that

$$n(\varepsilon, d) \ge c(1+\gamma)^d\,, \quad \text{for all } 0 < \varepsilon \le \varepsilon_0 \text{ and infinitely many } d \in \mathbb{N}\,,$$

then the problem suffers from *the curse of dimensionality*.
We need a further notion of tractability, namely *quasi-polynomial tractability*, see for instance [8]. In fact, the approximation problem is called quasi-polynomially tractable if there are positive numbers t and C_t such that

$$n(\varepsilon, d) \le C_t \exp(t \ln(\varepsilon^{-1})(1 + \ln(d)))\,. \tag{5.2}$$

Of course, quasi-polynomial tractability implies weak tractability.

5.2 Tractability results for $H^s_{\text{mix}}(\mathbb{T}^d)$

By our results in Section 4 we are very well prepared for the investigation of these tractability problems, resulting in short proofs of the assertions.

Theorem 5.1. *For every $s > 0$ the approximation problem for the embeddings*

$$I_d : H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$$

is quasi-polynomially tractable.

Proof. Let $0 < \varepsilon \le 1$ be given. Select $r \in \mathbb{N}$ such that $r^{-s} < \varepsilon \le (r-1)^{-s}$. Employing Lemma 4.1 we get $a_{C(r,d)}(I_d : H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = 1/r^s < \varepsilon$. This implies

$$n(\varepsilon, d) \le n(r^{-s}, d) \le C(r, d)\,.$$

Once more, the problem reduces to the estimation of $C(r, d)$. Parts of it have already been done in the proof of Theorem 4.9 above. There we observed that $C(r, d) \le e^2 r^{2+\log_2 d}$ if $r \le 2^d$. In case $r \ge e^d$ we proved $C(r, d) \le r^2\, 3^d \le r^4$, see (4.7). It remains to deal with $2^d < r < e^d$. Based on Lemma 3.1, Lemma 3.3 and taking into account $x^k/k! \le e^x$ applied to $x = \ln r$ we conclude

$$C(r, d) \le 1 + \sum_{\ell=1}^{d} \binom{d}{\ell} 2^\ell\, r\, \frac{(\ln r)^{\ell-1}}{(\ell-1)!} \le r^2 \sum_{\ell=1}^{d} \binom{d}{\ell} 2^\ell \le 3^d\, r^2 \le r^4\,.$$

This gives

$$\ln(n(\varepsilon, d)) \le \ln C(r, d) \le \begin{cases} 2 + (2 + \log_2 d) \ln r & : \quad r \le 2^d\,, \\ 4 \ln r & : \quad r \ge 2^d\,. \end{cases}$$

Since $\ln r \sim \ln(1/\varepsilon)/s$ we obtain (5.2). The proof is complete. ■

Corollary 5.2. *For every $s > 0$ and every $m \in \mathbb{N}$ the approximation problems for the embeddings*

$$I_d : H^{s,+}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d) \quad , \quad I_d : H^{s,*}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d) \quad \textit{and} \quad I_d : H^{m}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$$

are quasi-polynomially tractable.

Proof. The results are direct consequences of the embeddings in Lemma 2.2. Because of Lemma 2.2/(iv) it is sufficient to consider $s < 1/2$. Then Lemma 2.2/(iii) and Theorem 5.1 immediately imply the statement for $H^{s,*}_{\text{mix}}(\mathbb{T}^d)$. Finally, Lemma 2.2/(v) and Theorem 5.1 give quasi-polynomial tractability for $H^{s,+}_{\text{mix}}(\mathbb{T}^d)$. ■

Remark 5.3. Tensor product problems play an essential role in IBC (information based complexity), see, e.g., Section 2 in Chapter 5 and Section 2 in Chapter 8 of the monograph [17]. The spaces $H^m_{\text{mix}}(\mathbb{T}^d)$, $H^{s,\#}_{\text{mix}}(\mathbb{T}^d)$, $H^{s,+}_{\text{mix}}(\mathbb{T}^d)$, and $H^{s,*}_{\text{mix}}(\mathbb{T}^d)$ are d-fold tensor products of the univariate spaces $H^m(\mathbb{T})$, $H^{s,\#}(\mathbb{T})$, $H^{s,+}(\mathbb{T})$, and $H^{s,*}(\mathbb{T})$, respectively, see for instance [26]. Obviously, the identity I_d is a compact tensor product operator (considered as a mapping into the tensor product space $L_2(\mathbb{T}^d)$). Since the approximation numbers decay polynomially in these four univariate situations and $1 = a_1 > a_2$, we obtain the following conclusion from the general Theorem 3.3 of [8]: For any $s > 0$, all four problems $I_d : H^m_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$, $I_d : H^{s,\#}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$, $I_d : H^{s,+}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$, and $I_d : H^{s,*}_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$ are quasi-polynomially tractable (and polynomially intractable). Let us also mention, that this result in [8] has a forerunner in [18], where it has been proven that such tensor product problems are weakly tractable. Hence, Theorem 5.1 and Corollary 5.2 are special cases of a more general theory. However, the approach given here is different.

Acknowledgments. The authors would like to thank the organizers of the Dagstuhl seminar 12391 "Algorithms and Complexity for Continuous Problems", 2012, where this work has been initiated, for providing a pleasant and fruitful working atmosphere. In addition, the authors would like to thank Jan Vybíral and Glenn Byrenheid for commenting on earlier versions of this manuscript.

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