Efficient quantum algorithm for dissipative nonlinear differential equations

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Abstract

While there has been extensive previous work on efficient quantum algorithms for linear differential equations, analogous progress for *nonlinear* differential equations has been severely limited due to the linearity of quantum mechanics. Despite this obstacle, we develop a quantum algorithm for initial value problems described by dissipative quadratic n-dimensional ordinary differential equations. Assuming $\mathrm{R} < 1$, where R is a parameter characterizing the ratio of the nonlinearity to the linear dissipation, this algorithm has complexity $T^2\,\mathrm{poly}(\log T, \log n)/\epsilon$, where T is the evolution time and ϵ is the allowed error in the output quantum state. This is an exponential improvement over the best previous quantum algorithms, whose complexity is exponential in T. We achieve this improvement using the method of Carleman linearization, for which we give an improved convergence theorem. This method maps a system of nonlinear differential equations to an infinite-dimensional system of linear differential equations, which we discretize, truncate, and solve using the forward Euler method and the quantum linear system algorithm. We also provide a lower bound on the worst-case complexity of quantum algorithms for general quadratic differential equations, showing that the problem is intractable for $\mathrm{R} \geq \sqrt{2}$. Finally, we discuss potential applications of this approach to problems arising in biology as well as in fluid and plasma dynamics.

1 Introduction

Models governed by both ordinary differential equations (ODEs) and partial differential equations (PDEs) arise extensively in natural and social science, medicine, and engineering. Such equations characterize physical and biological systems that exhibit a wide variety of complex phenomena, including turbulence and chaos.

We focus here on differential equations with nonlinearities that can be expressed with quadratic polynomials. Note that polynomials of degree higher than two, and even more general nonlinearities, can be reduced to the quadratic case by introducing additional variables [30, 35]. The quadratic case also directly includes many archetypal models, such as the logistic equation in biology, the Lorenz system in atmospheric dynamics, and the Navier–Stokes equations in fluid dynamics.

Quantum algorithms offer the prospect of rapidly characterizing the solutions of high-dimensional systems of linear ODEs [8, 10, 19] and PDEs [15, 20, 22, 23, 28, 40, 42]. Such algorithms can produce a quantum state proportional to the solution of a sparse (or block-encoded) n-dimensional system of linear differential equations in time $\mathrm{poly}(\log n)$ using the quantum linear system algorithm [33].

Early work on quantum algorithms for differential equations already considered the nonlinear case [39]. It gave a quantum algorithm for ODEs that simulates polynomial nonlinearities by storing multiple copies of the solution. The complexity of this approach is polynomial in the logarithm of the dimension but exponential in the evolution time, owing to exponentially increasing resources used to maintain sufficiently many copies of the solution to represent the nonlinearity throughout the evolution.

Recently, heuristic quantum algorithms for nonlinear ODEs have been studied. Reference [34] explores a linearization technique known as the Koopman–von Neumann method that might be amenable to the quantum linear system algorithm. In [27], the authors provide a high-level description of how linearization can help solve nonlinear equations on a quantum computer. However, neither paper makes precise statements about concrete implementations or running times of quantum algorithms.

While quantum mechanics is described by linear dynamics, possible nonlinear modifications of the theory have been widely studied. Generically, such modifications enable quickly solving hard computational problems (e.g., solving unstructured search among n items in time $\mathrm{poly}(\log n)$), making nonlinear dynamics exponentially difficult to simulate in general [1, 2, 21]. Therefore, constructing efficient quantum algorithms for general classes of nonlinear dynamics has been considered largely out of reach.

In this article, we design and analyze a quantum algorithm that overcomes this limitation using Carleman linearization [16, 30, 37]. This approach embeds polynomial nonlinearities into an infinite-dimensional system of linear ODEs, and then truncates it to obtain a finite-dimensional linear approximation. The Carleman method has previously been used in the analysis of dynamical systems [3, 41, 46] and the design of control systems [13, 31, 44], but to the best of our knowledge it has not been employed in the context of quantum algorithms. We discretize the finite ODE system in time using the forward Euler method and solve the resulting linear equations with the quantum linear system algorithm [18, 33]. We control the approximation error of this approach by combining an improved version of a convergence theorem in [30] with a bound for the global error of the Euler method. Furthermore, we provide an upper bound for the condition number of the linear system and lower bound the success probability of the final measurement. Subject to the condition $\mathrm{R} < 1$, where the quantity R (defined in Problem 1 below) characterizes the relative strength of the nonlinear and dissipative linear terms, we show that the total complexity of this *quantum Carleman linearization algorithm* is $T^2\,\mathrm{poly}(\log T, \log n)/\epsilon$, where T is the evolution time, n is the dimension, and ϵ is the allowed error (Theorem 1). In the regime $\mathrm{R} < 1$, this is an exponential improvement over [39], which has complexity exponential in T.

We also provide a quantum lower bound for the worst-case complexity of simulating strongly nonlinear dynamics, showing that the algorithm's condition $\mathrm{R} < 1$ cannot be significantly improved in general (Theorem 2). Following the approach of [2, 21], we construct a protocol for distinguishing two states of a qubit driven by a certain quadratic ODE. Provided $\mathrm{R} \geq \sqrt{2}$, this procedure distinguishes states with overlap $1 - \epsilon$ in time $\mathrm{poly}(\log(1/\epsilon))$. Since state distinguishability can be used to solve the unstructured search problem, the quantum search lower bound [7] thereby implies a lower bound on the complexity of the quantum ODE problem.

Our quantum algorithm could potentially be applied to study models governed by quadratic ODEs arising in biology and epidemiology as well as in fluid and plasma dynamics. In particular, the celebrated Navier–Stokes equation with linear damping, which describes many physical phenomena,

can be treated by our approach provided the Reynolds number is sufficiently small. We also note that while the formal validity of our arguments assumes $R < 1$, we find in one numerical experiment that our proposed approach remains valid for larger R (see Section 6).

The remainder of this paper is structured as follows. Section 2 introduces the quantum quadratic ODE problem. Section 3 presents the Carleman linearization procedure and describes its performance. Section 4 gives a detailed analysis of the quantum Carleman linearization algorithm. Section 5 establishes a quantum lower bound for simulating quadratic ODEs. Section 6 describes how our approach could be applied to several well-known ODEs and PDEs and presents numerical results for the case of the viscous Burgers equation. Finally, we conclude with a discussion of the results and some possible future directions in Section 7.

2 Quadratic ODEs

We focus on an initial value problem described by the n-dimensional quadratic ODE

$$\frac{\mathrm{d}u}{\mathrm{d}t} = F_1 u + F_2 u^{\otimes 2}, \qquad u(0) = u_{\mathrm{in}}. \tag{2.1}$$

Here $u = [u_1, \ldots, u_n]^T \in \mathbb{R}^n$, $u^{\otimes 2} = [u_1^2, u_1u_2, \ldots, u_1u_n, u_2u_1, \ldots, u_nu_{n-1}, u_n^2]^T \in \mathbb{R}^{n^2}$, each $u_j = u_j(t)$ is a function of t on the interval $[0, T]$ for $j \in [n] \coloneqq \{1, \ldots, n\}$, and $F_1 \in \mathbb{R}^{n\times n}$, $F_2 \in \mathbb{R}^{n\times n^2}$ are time-independent matrices. We let $\|\cdot\|$ denote the spectral norm.

The main computational problem we consider is as follows.

Problem 1. *In the* quantum quadratic ODE problem*, we consider an n-dimensional quadratic ODE as in (2.1). We assume F_1 and F_2 are s-sparse (i.e., have at most s nonzero entries in each row and column), F_1 is diagonalizable, and that the eigenvalues λ_j of F_1 satisfy $\mathrm{Re}\,(\lambda_n) \leq \cdots \leq \mathrm{Re}\,(\lambda_1) < 0$. We parametrize the problem in terms of the quantity*

$$\mathrm{R} \coloneqq \frac{\|u_{\mathrm{in}}\|\|F_2\|}{|\,\mathrm{Re}\,(\lambda_1)|}. \tag{2.2}$$

We assume the values $\mathrm{Re}\,(\lambda_1)$, $\|F_1\|$, and $\|F_2\|$ are known, and that we are given oracles O_{F_1} and O_{F_2} that provide the locations and values of the nonzero entries of F_1 and F_2, respectively, for any desired row or column. We are also given the value $\|u_{\mathrm{in}}\|$ and an oracle O_x that maps $|00\ldots0\rangle \in \mathbb{C}^n$ to a quantum state proportional to u_{in}. Our goal is to produce a quantum state proportional to $u(T)$ for some given $T > 0$ within some prescribed error tolerance $\epsilon > 0$.

The quantity R is qualitatively similar to the Reynolds number, which characterizes the ratio of the (nonlinear) convective forces to the (linear) viscous forces within a fluid [14, 38].

Note that without loss of generality, given a quadratic ODE satisfying (2.1) with $R < 1$, we can modify it by rescaling $u \to \gamma u$ with a suitable constant γ to satisfy

$$\|F_2\| < |\mathrm{Re}\,(\lambda_1)| \tag{2.3}$$

and

$$\|u_{\mathrm{in}}\| \leq 1, \tag{2.4}$$

with R left unchanged by the rescaling. We use this rescaling in our algorithm and its analysis. With this rescaling, a small R implies either small $\|u_{\mathrm{in}}\|$ and/or small $\|F_2\|$ relative to $|\mathrm{Re}\,(\lambda_1)|$.

3 Quantum Carleman linearization

As discussed in Section 1, it is challenging to directly simulate quadratic ODEs using quantum computers, and indeed the complexity of the best known quantum algorithm is exponential in the evolution time T [39]. However, for a dissipative nonlinear ODE without a source, any quadratic nonlinear effect will only be significant for a finite time because of the dissipation. To exploit this, we can create a linear system that approximates the initial nonlinear evolution within some controllable error. After the nonlinear effects are no longer important, the linear system properly captures the almost-linear evolution from then on.

We develop such a quantum algorithm using the concept of Carleman linearization [16, 30, 37]. Carleman linearization is a method for converting a finite-dimensional system of nonlinear differential equations into an infinite-dimensional linear one. This is achieved by introducing powers of the variables into the system, allowing it to be written as an infinite sequence of coupled linear differential equations. We then truncate the system to N equations, where the truncation level N depends on the allowed approximation error, giving a finite linear ODE system.

Let us describe the Carleman linearization procedure in more detail. Given a system of quadratic ODEs (2.1), we apply the Carleman procedure to obtain the system of linear ODEs

$$\frac{\mathrm{d}\hat{y}}{\mathrm{d}t} = A\hat{y}, \qquad \hat{y}(0) = \hat{y}_{\mathrm{in}} \tag{3.1}$$

with the upper triangular block structure

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \vdots \\ \hat{y}_{N-1} \\ \hat{y}_N \end{pmatrix} = \begin{pmatrix} A_1^1 & A_2^1 & & & & \\ & A_2^2 & A_3^2 & & & \\ & & A_3^3 & A_4^3 & & \\ & & & \ddots & \ddots & \\ & & & & A_{N-1}^{N-1} & A_N^{N-1} \\ & & & & & A_N^N \end{pmatrix} \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \vdots \\ \hat{y}_{N-1} \\ \hat{y}_N \end{pmatrix}, \tag{3.2}$$

where $\hat{y}_j = u^{\otimes j} \in \mathbb{R}^{n^j}$, $\hat{y}_{\mathrm{in}} = [u_{\mathrm{in}}; u_{\mathrm{in}}^{\otimes 2}; \ldots; u_{\mathrm{in}}^{\otimes N}]$, and $A_j^j \in \mathbb{R}^{n^j \times n^j}$, $A_{j+1}^j \in \mathbb{R}^{n^j \times n^{j+1}}$ for $j \in [N]$ satisfying

$$A_j^j = F_1 \otimes I^{\otimes j-1} + I \otimes F_1 \otimes I^{\otimes j-2} + \cdots + I^{\otimes j-1} \otimes F_1, \tag{3.3}$$

$$A_{j+1}^j = F_2 \otimes I^{\otimes j-1} + I \otimes F_2 \otimes I^{\otimes j-2} + \cdots + I^{\otimes j-1} \otimes F_2. \tag{3.4}$$

Note that A is an (Ns)-sparse matrix. The dimension of (3.1) is

$$\Delta := n + n^2 + \cdots + n^N = \frac{n^{N+1} - n}{n-1} = O(n^N). \tag{3.5}$$

To construct a system of linear equations, we next divide $[0, T]$ into $m = T/h$ time steps and apply the forward Euler method on (3.1), letting

$$\frac{y^{k+1} - y^k}{h} = Ay^k \tag{3.6}$$

where $y^k \in \mathbb{R}^\Delta$ approximates $\hat{y}(kh)$ for each $k \in [m+1]_0 := \{0, 1, \ldots, m\}$, with $y^0 = y_{\mathrm{in}} := \hat{y}(0) = \hat{y}_{\mathrm{in}}$, and letting all y^k be equal for $k \in [m+p+1]_0 \setminus [m+1]_0$, for some sufficiently large integer p. This gives an $(m+p+1)\Delta \times (m+p+1)\Delta$ linear system

$$L|Y\rangle = |B\rangle \tag{3.7}$$

that encodes (3.6) and uses it to produce a numerical solution at time T, where

$$L = \sum_{k=0}^{m+p} |k\rangle\langle k| \otimes I - \sum_{k=1}^{m} |k\rangle\langle k-1| \otimes (I + Ah) - \sum_{h=m+1}^{m+p} |k\rangle\langle k-1| \otimes I. \tag{3.8}$$

Observe that the system (3.7) has the lower triangular structure

$$\begin{pmatrix} I & & & & & & & \\ -(I+Ah) & I & & & & & & \\ & \ddots & \ddots & & & & & \\ & & -(I+Ah) & I & & & & \\ & & & -I & I & & & \\ & & & & \ddots & \ddots & & \\ & & & & & -I & I \end{pmatrix} \begin{pmatrix} y^0 \\ y^1 \\ \vdots \\ y^m \\ y^{m+1} \\ \vdots \\ y^{m+p} \end{pmatrix} = \begin{pmatrix} y_{\text{in}} \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.9}$$

In the above system, the first n components of y^k for $k \in [m+p+1]_0$ (i.e., y_1^k) approximate the exact solution $u(T)$, up to normalization. We apply the high-precision quantum linear system algorithm (QLSA) [18] to (3.7) and postselect on k to produce $y_1^k/\|y_1^k\|$ for some $k \in [m+p+1]_0 \setminus [m]_0$. The resulting error is at most

$$\epsilon := \max_{k \in [m+p+1]_0 \setminus [m]_0} \left\| \frac{u(T)}{\|u(T)\|} - \frac{y_1^k}{\|y_1^k\|} \right\|. \tag{3.10}$$

This error includes contributions from both Carleman linearization and the forward Euler method. (The QLSA also introduces error, which we bound separately. Note that we could instead apply the original QLSA [33] instead of its subsequent improvement [18], but this would slightly complicate the error analysis and might perform worse in practice.)

Now we state our main algorithmic result.

Theorem 1 (Quantum Carleman linearization algorithm)**.** *Consider an instance of the quantum quadratic ODE problem as defined in Problem 1, with its Carleman linearization as defined in (3.1). Assume* $\mathrm{R} < 1$*. Let*

$$g := \|u(T)\|, \qquad\qquad q := \frac{\|u_{\text{in}}\|}{\|u(T)\|}. \tag{3.11}$$

There exists a quantum algorithm producing a state that approximates $u(T)/\|u(T)\|$ *with error at most* $\epsilon \le 1$*, succeeding with probability* $\Omega(1)$*, with a flag indicating success, using at most*

$$\frac{sT^2 q(\|F_1\| + \|F_2\|)^2}{(1-\mathrm{R})\|F_2\| g\epsilon} \cdot \mathrm{poly}\left(\log \frac{sT\|F_1\|\|F_2\| \log(1/R)}{(1-\mathrm{R}) g\epsilon} \right) \tag{3.12}$$

queries to the oracles O_{F_1}*,* O_{F_2}*, and* O_x*. The gate complexity is larger than the query complexity by a factor of* $\mathrm{poly}(\log(nsT\|F_1\|\|F_2\| \log(1/R)/((1-\mathrm{R})g\epsilon)))$*. Furthermore, if the eigenvalues of* F_1 *are all real, the query complexity is*

$$\frac{sT^2 q(\|F_1\| + \|F_2\|)^2}{g\epsilon} \cdot \mathrm{poly}\left(\log \frac{sT\|F_1\|\|F_2\| \log(1/R)}{g\epsilon} \right) \tag{3.13}$$

and the gate complexity is larger by a factor of $\mathrm{poly}(\log(nsT\|F_1\|\|F_2\| \log(1/R)/((1-\mathrm{R})g\epsilon)))$.

4 Algorithm analysis

In this section we establish several lemmas and use them to prove Theorem 1.

4.1 Solution error

The solution error has three contributions: the error from applying Carleman linearization to (2.1), the error in the time discretization of (3.1) by the forward Euler method, and the error from the QLSA. Since the QLSA produces a solution with error at most ϵ with complexity $\mathrm{poly}(\log(1/\epsilon))$ [18], we focus on bounding the first two contributions.

4.1.1 Error from Carleman linearization

First, we provide an upper bound for the error from Carleman linearization for arbitrary evolution time T. To the best of our knowledge, the first and only explicit bound on the error of Carleman linearization appears in [30]. However, to bound the error for arbitrary T, they assume the logarithmic norm of F_1 is negative (see Theorems 4.2 and 4.3 of [30]), which is too strong for our case. Instead, we give an improved analysis under milder conditions. This analysis follows the proof in [30] closely.

We begin with a lemma that describes the decay of the solution of (2.1).

Lemma 1. *Consider an instance of the quadratic ODE (2.1), and assume* $\mathrm{R} < 1$ *as defined in (2.2). Then the solution* $u(t)$ *of (2.1) satisfies* $\|u(t)\| < \|u_{\mathrm{in}}\|$ *for any* $t > 0$.

Proof. Consider the derivative of $\|u(t)\|$. We have

$$\begin{aligned}\frac{d\|u\|^2}{dt} &= u^\dagger(F_1 + F_1^\dagger)u + u^\dagger F_2(u\otimes u) + (u^\dagger \otimes u^\dagger)F_2^\dagger u\,,\\ &\leq 2\,\mathrm{Re}(\lambda_1)\|u\|^2 + 2\|u\|\|F_2\|\|u\|^2\,.\end{aligned} \tag{4.1}$$

If $\|u\| \neq 0$, then

$$\frac{d\|u\|}{dt} \leq \mathrm{Re}(\lambda_1)\|u\| + \|F_2\|\|u\|^2\,. \tag{4.2}$$

Solving the differential inequality as an equation gives us a bound on $\|u\|$. The differential equation can be solved using standard techniques (see [30] for a solution). Letting $\mathrm{Re}(\lambda_1) = -a$ (and assuming $a > 0$) and $\|F_2\| = b$, we get

$$\|u\| \leq \frac{a}{b - e^{at}(b - a/\|u_{\mathrm{in}}\|)}\,. \tag{4.3}$$

This shows that the norm decreases if $b - a/\|u_{\mathrm{in}}\| < 0$, which is the same as $\mathrm{R} = \|u_{\mathrm{in}}\|b/a < 1$. □

We now give an upper bound on the error of Carleman linearization.

Lemma 2. *Consider an instance of the quadratic ODE (2.1), with its corresponding Carleman linearization as defined in (3.1). As in Problem 1, assume that the eigenvalues* λ_j *of* F_1 *satisfy* $\mathrm{Re}\,(\lambda_n) \leq \cdots \leq \mathrm{Re}\,(\lambda_1) < 0$. *Then for any* $j \in [N]$, *the error* $\eta_j(t) \coloneqq u^{\otimes j}(t) - \hat{y}_j(t)$ *satisfies*

$$\|\eta_j(t)\| \leq \|u_{\mathrm{in}}\|^j\,\mathrm{R}^{N+1-j} \tag{4.4}$$

where we require $\mathrm{R} < 1$ *as defined in (2.2). For* $j = 1$, *we have the tighter bound*

$$\|\eta_1(t)\| \leq \|u_{\mathrm{in}}\|\,\mathrm{R}^N\left(1 - e^{\mathrm{Re}(\lambda_1)t}\right)^N. \tag{4.5}$$

Proof. The exact solution $u(t)$ of the original quadratic ODE (2.1) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ u^{\otimes 2} \\ u^{\otimes 3} \\ \vdots \\ u^{\otimes(N-1)} \\ u^{\otimes N} \\ \vdots \end{pmatrix} = \begin{pmatrix} A_1^1 & A_2^1 & & & & & \\ & A_2^2 & A_3^2 & & & & \\ & & A_3^3 & A_4^3 & & & \\ & & & \ddots & \ddots & & \\ & & & & A_{N-1}^{N-1} & A_N^{N-1} & \\ & & & & & A_N^N & \ddots \\ & & & & & & \ddots \end{pmatrix} \begin{pmatrix} u \\ u^{\otimes 2} \\ u^{\otimes 3} \\ \vdots \\ u^{\otimes(N-1)} \\ u^{\otimes N} \\ \vdots \end{pmatrix}, \tag{4.6}$$

and the approximated solution $\hat{y}_j(t)$ satisfies (3.2). Comparing these equations, we have

$$\frac{\mathrm{d}\eta_j}{\mathrm{d}t} = A_j^j \eta_j + A_{j+1}^j \eta_{j+1}, \quad j \in [N-1] \tag{4.7}$$

and

$$\frac{\mathrm{d}\eta_N}{\mathrm{d}t} = A_N^N \eta_N + A_{N+1}^N u^{\otimes(N+1)}. \tag{4.8}$$

We proceed by backward substitution. Since $\eta_N(0) = 0$, we have

$$\eta_N(t) = \int_0^t e^{A_N^N (t-s_0)} A_{N+1}^N u^{\otimes(N+1)}(s_0)\,\mathrm{d}s_0. \tag{4.9}$$

For $j \in [N]$, (3.3) gives $\|e^{A_j^j t}\| = e^{j\,\mathrm{Re}(\lambda_1)t}$ and (3.4) gives $\|A_{j+1}^j\| = j\|F_2\|$. By Lemma 1, $\|u^{\otimes(N+1)}\| = \|u\|^{N+1} \le \|u_{\mathrm{in}}\|^{N+1}$. We can therefore upper bound (4.9) by

$$\begin{aligned} \|\eta_N(t)\| &\le \int_0^t \|e^{A_N^N (t-s_0)}\| \cdot \|A_{N+1}^N u^{\otimes(N+1)}(s_0)\|\,\mathrm{d}s_0 \\ &\le N\|F_2\|\|u_{\mathrm{in}}\|^{N+1} \int_0^t e^{N\,\mathrm{Re}(\lambda_1)(t-s_0)}\,\mathrm{d}s_0. \end{aligned} \tag{4.10}$$

For $j = N-1$, we have

$$\frac{\mathrm{d}\eta_{N-1}}{\mathrm{d}t} = A_{N-1}^{N-1}\eta_{N-1} + A_N^{N-1}\eta_N. \tag{4.11}$$

Again, since $\eta_{N-1}(0) = 0$, we have

$$\eta_{N-1}(t) = \int_0^t e^{A_{N-1}^{N-1}(t-s_1)} A_N^{N-1} \eta_N(s_1)\,\mathrm{d}s_1 \tag{4.12}$$

which has the upper bound

$$\begin{aligned} \|\eta_{N-1}(t)\| &\le \int_0^t \|e^{A_{N-1}^{N-1}(t-s_1)}\| \cdot \|A_N^{N-1}\eta_N(s_1)\|\,\mathrm{d}s_1 \\ &\le (N-1)\|F_2\| \int_0^t e^{(N-1)\,\mathrm{Re}(\lambda_1)(t-s_1)} \|\eta_N(s_1)\|\,\mathrm{d}s_1 \\ &\le N(N-1)\|F_2\|^2\|u_{\mathrm{in}}\|^{N+1} \int_0^t e^{(N-1)\,\mathrm{Re}(\lambda_1)(t-s_1)} \int_0^{s_1} e^{N\,\mathrm{Re}(\lambda_1)(s_1-s)}\,\mathrm{d}s_0\,\mathrm{d}s_1, \end{aligned} \tag{4.13}$$

where we used (4.10) in the last step. Iterating this procedure for $j = N-2, \dots, 1$, we find

$$\begin{aligned}\|\eta_j(t)\| &\le \frac{N!}{(j-1)!}\|F_2\|^{N+1-j}\|u_{\text{in}}\|^{N+1}\int_0^t e^{j\,\mathrm{Re}(\lambda_1)(t-s_{N-j})}\int_0^{s_{N-j}} e^{(j+1)\,\mathrm{Re}(\lambda_1)(s_{N-j}-s_{N-1-j})}\cdots \\ &\quad \int_0^{s_2} e^{(N-1)\,\mathrm{Re}(\lambda_1)(s_2-s_1)}\int_0^{s_1} e^{N\,\mathrm{Re}(\lambda_1)(s_1-s_0)}\,\mathrm{d}s_0\,\mathrm{d}s_1\cdots\mathrm{d}s_{N-j-1}\,\mathrm{d}s_{N-j} \\ &= \frac{N!}{(j-1)!}\|F_2\|^{N+1-j}\|u_{\text{in}}\|^{N+1}\int_0^{s_{N+1-j}}\cdots\int_0^{s_2}\int_0^{s_1} e^{\mathrm{Re}(\lambda_1)(-Ns_0+\sum_{k=1}^{N-j+1}s_k)}\,\mathrm{d}s_0\,\mathrm{d}s_1\cdots\mathrm{d}s_{N-j}.\end{aligned} \tag{4.14}$$

Finally, using

$$\int_0^{s_{k+1}} e^{(N-k)\,\mathrm{Re}(\lambda_1)(s_{k+1}-s_k)}\mathrm{d}s_k = \frac{1-e^{(N-k)\,\mathrm{Re}(\lambda_1)s_{k+1}}}{(N-k)|\mathrm{Re}(\lambda_1)|} \le \frac{1}{(N-k)|\mathrm{Re}(\lambda_1)|} \tag{4.15}$$

for $k = 0, \dots, N-j$, we have

$$\|\eta_j(t)\| \le \frac{N!}{(j-1)!}\|F_2\|^{N+1-j}\|u_{\text{in}}\|^{N+1}\frac{(j-1)!}{N!|\mathrm{Re}(\lambda_1)|^{N+1-j}} = \frac{\|u_{\text{in}}\|^{N+1}\|F_2\|^{N+1-j}}{|\mathrm{Re}(\lambda_1)|^{N+1-j}} = \|u_{\text{in}}\|^j\,\mathrm{R}^{N+1-j}. \tag{4.16}$$

For $j = 1$, the bound can be further improved. By Lemma 5.2 of [30], if $a \neq 0$,

$$\int_0^{s_N}\cdots\int_0^{s_2}\int_0^{s_1} e^{a(-Ns_0+\sum_{k=1}^N s_k)}\,\mathrm{d}s_0\,\mathrm{d}s_1\cdots\mathrm{d}s_{N-1} = \frac{(e^{as_N}-1)^N}{N!a^N}. \tag{4.17}$$

With $s_N = t$ and $a = \mathrm{Re}(\lambda_1)$, we find

$$\begin{aligned}\|\eta_1(t)\| &\le N!\|F_2\|^N\|u_{\text{in}}\|^{N+1}\int_0^{s_N}\cdots\int_0^{s_2}\int_0^{s_1} e^{\mathrm{Re}(\lambda_1)(-Ns_0+\sum_{k=1}^N s_k)}\,\mathrm{d}s_0\,\mathrm{d}s_1\cdots\mathrm{d}s_{N-1} \\ &\le N!\|F_2\|^N\|u_{\text{in}}\|^{N+1}\frac{(e^{\mathrm{Re}(\lambda_1)t}-1)^N}{N!\,\mathrm{Re}(\lambda_1)^N} \\ &= \|u_{\text{in}}\|\,\mathrm{R}^N\left(1-e^{at}\right)^N,\end{aligned} \tag{4.18}$$

which is tighter than the $j = 1$ case in (4.16). □

While Problem 1 makes some strong assumptions about the system of differential equations, they appear to be necessary for our analysis. Specifically, the conditions $\mathrm{Re}\,(\lambda_1) < 0$ and $\mathrm{R} < 1$ are required to ensure arbitrary-time convergence.

Since the Euler method for (3.1) is unstable if $\mathrm{Re}\,(\lambda_1) > 0$ [8, 25], we only consider the case $\mathrm{Re}\,(\lambda_1) \le 0$. If $\mathrm{Re}\,(\lambda_1) = 0$, (4.17) reduces to

$$\int_0^{s_N}\cdots\int_0^{s_2}\int_0^{s_1} e^{a(-Ns_0+\sum_{k=1}^N s_k)}\,\mathrm{d}s_0\,\mathrm{d}s_1\cdots\mathrm{d}s_{N-1} = \frac{t^N}{N!}, \tag{4.19}$$

giving the error bound

$$\|\eta_1(t)\| \le \|u_{\text{in}}\|(\|u_{\text{in}}\|\|F_2\|t)^N \tag{4.20}$$

instead of (4.18). Then the error bound can be made arbitrarily small for a finite time by increasing N, but after $t > 1/\|u_{\text{in}}\|\|F_2\|$, the error bound diverges.

Furthermore, if $\mathrm{R} \ge 1$, Bernoulli's inequality gives

$$\|u_{\text{in}}\|(1-Ne^{\mathrm{Re}(\lambda_1)t}) \le \|u_{\text{in}}\|(1-Ne^{\mathrm{Re}(\lambda_1)t})\,\mathrm{R}^N \le \|u_{\text{in}}\|(1-e^{\mathrm{Re}(\lambda_1)t})^N\,\mathrm{R}^N, \tag{4.21}$$

where the right-hand side upper bounds $\|\eta_1(t)\|$ as in (4.18). Assuming $\|\eta_1(t)\| = \|u(t) - \hat{y}_1(t)\|$ is smaller than $\|u_{\text{in}}\|$, we require

$$N = \Omega(e^{|\operatorname{Re}(\lambda_1)|t}). \tag{4.22}$$

In other words, to apply (4.5) for the Carleman linearization procedure, the truncation order given by Lemma 2 must grow exponentially with t.

In fact, we prove in Section 5 that for $\mathrm{R} \geq \sqrt{2}$, *no* quantum algorithm (even one based on a technique other than Carleman linearization) can solve Problem 1 efficiently. It remains open to understand the complexity of the problem for $1 \leq \mathrm{R} < \sqrt{2}$.

On the other hand, if $\mathrm{R} < 1$, both (4.4) and (4.5) decrease exponentially with N, making the truncation efficient. We discuss the specific choice of N in (4.96) below.

4.1.2 Error from forward Euler method

Next, we provide an upper bound for the error incurred by approximating (3.1) with the forward Euler method. This problem has been well studied for general ODEs. Given an ODE $\frac{\mathrm{d}z}{\mathrm{d}t} = f(z)$ on $[0, T]$ with an inhomogenity f that is an L-Lipschitz continuous function of z, the global error of the solution is upper bounded by e^{LT}, although in most cases this bound overestimates the actual error [4]. To remove the exponential dependence on T in our case, we derive a tighter bound for time discretization of (3.1) in Lemma 3 below. This lemma is potentially useful for other ODEs as well and can be straightforwardly adapted to other problems.

Lemma 3. *Consider an instance of the quantum quadratic ODE problem as defined in Problem 1, with* $\mathrm{R} < 1$ *as defined in (2.2). Choose a time step*

$$h \leq \min\left\{ \frac{1}{N\|F_1\|}, \frac{2(|\operatorname{Re}(\lambda_1)| - \|F_2\|)}{N(|\operatorname{Re}(\lambda_1)|^2 - \|F_2\|^2 + \|F_1\|^2)} \right\} \tag{4.23}$$

in general, or

$$h \leq \frac{1}{N\|F_1\|} \tag{4.24}$$

if the eigenvalues of F_1 *are all real. Then the global error from the forward Euler method (3.6) on the interval* $[0, T]$ *for (3.1) satisfies*

$$\|\hat{y}_1(T) - y_1^m\| \leq \|\hat{y}(T) - y^m\| \leq N^{2.5} T h (\|F_1\| + \|F_2\|)^2. \tag{4.25}$$

Proof. We define a linear system that locally approximates the initial value problem (3.1) on $[kh, (k+1)h]$ for $k \in [m]_0$ as

$$z^k = (I + Ah)\hat{y}((k-1)h), \tag{4.26}$$

where $\hat{y}(t)$ is the exact solution of (3.1). For $k \in [m]$, we denote the local truncation error by

$$e^k := \|\hat{y}(kh) - z^k\| \tag{4.27}$$

and the global error by

$$g^k := \|\hat{y}(kh) - y^k\|, \tag{4.28}$$

where y^k in (3.6) is the numerical solution. Note that $g^m = \|\hat{y}(T) - y^m\|$.

For the local truncation error, we Taylor expand $\hat{y}((k-1)h)$ with a second-order Lagrange remainder, giving

$$\hat{y}(kh) = \hat{y}((k-1)h) + \hat{y}'((k-1)h)h + \frac{\hat{y}''(\xi)h^2}{2} \tag{4.29}$$

for some $\xi \in [(k-1)h, kh]$. Since $\hat{y}'((k-1)h) = A\hat{y}((k-1)h)$ by (3.1), we have

$$\hat{y}(kh) = (I + Ah)\hat{y}((k-1)h) + \frac{\hat{y}''(\xi)h^2}{2} = z^k + \frac{\hat{y}''(\xi)h^2}{2}, \tag{4.30}$$

and thus

$$e^k = \|\hat{y}(kh) - z^k\| = \|\hat{y}''(\xi)h\|\frac{h^2}{2} \le \frac{Mh^2}{2}. \tag{4.31}$$

By the triangle inequality,

$$g^k = \|\hat{y}(kh) - y^k\| \le \|\hat{y}(kh) - z^k\| + \|z^k - y^k\| \le e^k + \|z^k - y^k\|. \tag{4.32}$$

Since y^k and z^k are obtained by the same linear system with different right-hand sides, we have the upper bound

$$\|z^k - y^k\| = \|(I+Ah)[\hat{y}((k-1)h) - y^{k-1}]\| \le \|I+Ah\| \cdot \|\hat{y}((k-1)h) - y^{k-1}\| = \|I+Ah\|g^{k-1}. \tag{4.33}$$

In order to provide an upper bound for $\|I + Ah\|$, we write

$$I + Ah = H_1 + H_2 \tag{4.34}$$

where

$$H_1 = I + \sum_{j=1}^{N} |j\rangle\langle j| \otimes A_j^j h, \tag{4.35}$$

$$H_2 = \sum_{j=1}^{N-1} |j\rangle\langle j+1| \otimes A_{j+1}^j h. \tag{4.36}$$

We provide upper bounds separately for $\|H_1\|$ and $\|H_2\|$ and use the bound $\|I+Ah\| \le \|H_1\|+\|H_2\|$.

The eigenvalues of A_j^j consist of all j-term sums of the eigenvalues of F_1. More precisely, they are $\{\sum_{\ell\in[j]} \lambda_{\mathcal{I}_\ell^j}\}_{\mathcal{I}^j \in [n]^j}$ where $\{\lambda_\ell\}_{\ell\in[n]}$ are the eigenvalues of F_1 and $\mathcal{I}^j \in [n]^j$ is a j-tuple of indices. The eigenvalues of H_1 are thus $\{1 + h\sum_{\ell\in[j]} \lambda_{\mathcal{I}_\ell^j}\}_{\mathcal{I}^j\in[n]^j, j\in[N]}$. With $J := \max_{\ell\in[n]} |\operatorname{Im}(\lambda_\ell)|$, we have

$$\begin{aligned}\left|1 + h\sum_{\ell\in[j]} \lambda_{\mathcal{I}_\ell^j}\right|^2 &= \left|1 + h\sum_{\ell\in[j]} \operatorname{Re}(\lambda_{\mathcal{I}_\ell^j})\right|^2 + \left|h\sum_{\ell\in[j]} \operatorname{Im}(\lambda_{\mathcal{I}_\ell^j})\right|^2 \\ &\le 1 - 2Nh|\operatorname{Re}(\lambda_1)| + N^2h^2(|\operatorname{Re}(\lambda_1)|^2 + J^2)\end{aligned} \tag{4.37}$$

where we used $Nh|\operatorname{Re}(\lambda_1)| \le 1$ by (4.23). Therefore

$$\|H_1\| = \max_{j\in[N]} \max_{\mathcal{I}^j\in[n]^j} \left|1 + h\sum_{\ell\in[j]} \lambda_{\mathcal{I}_\ell^j}\right| \le \sqrt{1 - 2Nh|\operatorname{Re}(\lambda_1)| + N^2h^2(|\operatorname{Re}(\lambda_1)|^2 + J^2)}. \tag{4.38}$$

We also have

$$\|H_2\| = \left\|\sum_{j=1}^{N-1} |j\rangle\langle j+1| \otimes A_{j+1}^j h\right\| \le \max_{j\in[N]} \|A_{j+1}^j\|h \le N\|F_2\|h. \tag{4.39}$$

Using the bounds (4.38) and (4.39), we aim to select the value of h to ensure

$$\|I + Ah\| \le \|H_1\| + \|H_2\| \le 1. \tag{4.40}$$

The assumption (4.23) implies

$$h \le \frac{2(|\operatorname{Re}(\lambda_1)| - \|F_2\|)}{N(|\operatorname{Re}(\lambda_1)|^2 - \|F_2\|^2 + J^2)} \tag{4.41}$$

(note that the denominator is non-zero due to (2.3)). Then we have

$$\begin{aligned} & N^2h^2(|\operatorname{Re}(\lambda_1)|^2 - \|F_2\|^2 + J^2) \le 2Nh(|\operatorname{Re}(\lambda_1)| - \|F_2\|) \\ \implies \quad & 1 - 2Nh|\operatorname{Re}(\lambda_1)| + N^2h^2(|\operatorname{Re}(\lambda_1)|^2 + J^2) \le 1 - 2N\|F_2\|h + N^2\|F_2\|^2h^2 \\ \implies \quad & \sqrt{1 - 2Nh|\operatorname{Re}(\lambda_1)| + N^2h^2(|\operatorname{Re}(\lambda_1)|^2 + J^2)} \le 1 - N\|F_2\|h, \end{aligned} \tag{4.42}$$

so $\|I + Ah\| \le 1$ as claimed.

The choice (4.23) can be improved if an upper bound on J is known. In particular, if $J = 0$, (4.41) simplifies to

$$h \le \frac{2}{N(|\lambda_1| + \|F_2\|)}, \tag{4.43}$$

which is satisfied by (4.24) using $\|F_2\| < |\lambda_1| \le \|F_1\|$.

Using this in (4.33), we have

$$\|z^k - y^k\| \le \|I + Ah\|g^{k-1} \le g^{k-1}. \tag{4.44}$$

Plugging (4.44) into (4.32) iteratively, we find

$$g^k \le g^{k-1} + e^k \le g^{k-2} + e^{k-1} + e^k \le \cdots \le \sum_{j=1}^{k} e^j, \quad k \in [m+1]_0. \tag{4.45}$$

Using (4.31), this shows that the global error from the forward Euler method is bounded by

$$\|\hat{y}^1(T) - y_1^m\| \le \|\hat{y}(T) - y^m\| = g^m \le \sum_{j=1}^{m} e^j \le m\frac{Mh^2}{2} = \frac{MTh}{2}. \tag{4.46}$$

Finally, we remove the dependence on M. We have

$$M = \sup_{\tau\in[0,T]} \|\hat{y}''(\tau)\| = \sup_{\tau\in[0,T]} \|A^2\hat{y}(\tau)\| \le \|A\|^2 \sup_{\tau\in[0,T]} \|\hat{y}(\tau)\|. \tag{4.47}$$

We can upper bound the norm of A as

$$\|A\| = \left\| \sum_{j=1}^{N} |j\rangle\langle j| \otimes A_j^j + \sum_{j=1}^{N-1} |j\rangle\langle j+1| \otimes A_{j+1}^j \right\| \le N(\|F_1\| + \|F_2\|). \tag{4.48}$$

Using (4.4), $\|u\| \le \|u_{\text{in}}\| \le 1$, and $\mathrm{R} < 1$, we have

$$\begin{aligned} \|\hat{y}(t)\|^2 \le \sum_{j=1}^{N} \|\hat{y}_j(t)\|^2 = \sum_{j=1}^{N} \|u^{\otimes j}(t) - \eta_j(t)\|^2 \le 2\sum_{j=1}^{N} (\|u^{\otimes j}(t)\|^2 + \|\eta_j(t)\|^2) \\ \le 2\sum_{j=1}^{N} \left(\|u_{\text{in}}\|^{2j} + \|u_{\text{in}}\|^{2j}\, \mathrm{R}^{2N+2-2j} \right) \le 2\sum_{j=1}^{N} (1+1) \le 4N \end{aligned} \tag{4.49}$$

for all $t \in [0, T]$. Therefore

$$M \le \|A\|^2 \sup_{\tau\in[0,T]} \|\hat{y}(\tau)\| \le 2N^{2.5}(\|F_1\| + \|F_2\|)^2. \tag{4.50}$$

Thus, (4.46) gives

$$\|\hat{y}^1(T) - y_1^m\| \le N^{2.5}Th(\|F_1\| + \|F_2\|)^2 \tag{4.51}$$

as claimed. □

4.2 Condition number

Now we upper bound the condition number of the linear system.

Lemma 4. *Consider an instance of the quantum quadratic ODE problem as defined in Problem 1. Apply the forward Euler method (3.6) with time step (4.23) to the Carleman linearization (3.1). Then the condition number of the matrix L defined in (3.7) satisfies*

$$\kappa \leq 3(m+p+1). \tag{4.52}$$

Proof. We begin by upper bounding $\|L\|$. We write

$$L = L_1 + L_2 + L_3, \tag{4.53}$$

where

$$L_1 = \sum_{k=0}^{m+p} |k\rangle\langle k| \otimes I, \tag{4.54}$$

$$L_2 = -\sum_{k=1}^{m} |k\rangle\langle k-1| \otimes (I + Ah), \tag{4.55}$$

$$L_3 = -\sum_{k=m+1}^{m+p} |k\rangle\langle k-1| \otimes I. \tag{4.56}$$

Clearly $\|L_1\| = \|L_3\| = 1$. Furthermore, $\|L_2\| \leq \|I + Ah\| \leq 1$ by (4.40), which follows from the choice of time step (4.23). Therefore,

$$\|L\| \leq \|L_1\| + \|L_2\| + \|L_3\| \leq 3. \tag{4.57}$$

Next we upper bound

$$\|L^{-1}\| = \sup_{\||B\rangle\|\leq 1} \|L^{-1}|B\rangle\|. \tag{4.58}$$

We express $|B\rangle$ as

$$|B\rangle = \sum_{k=0}^{m+p} \beta_k |k\rangle = \sum_{k=0}^{m+p} |b^k\rangle, \tag{4.59}$$

where $|b^k\rangle \coloneqq \beta_k|k\rangle$ satisfies

$$\sum_{k=0}^{m+p} \||b^k\rangle\|^2 = \||B\rangle\|^2 \leq 1. \tag{4.60}$$

Given any $|b^k\rangle$ for $k \in [m+p+1]_0$, we define

$$|Y^k\rangle \coloneqq L^{-1}|b^k\rangle = \sum_{l=0}^{m+p} \gamma_l^k |l\rangle = \sum_{l=0}^{m+p} |Y_l^k\rangle, \tag{4.61}$$

where $|Y_l^k\rangle \coloneqq \gamma_l^k|l\rangle$. We first upper bound $\||Y^k\rangle\| = \|L^{-1}|b^k\rangle\|$, and then use this to upper bound $\|L^{-1}|B\rangle\|$.

We consider two cases. First, for fixed $k \in [m+1]_0$, we directly calculate $|Y_l^k\rangle$ for each $l \in [m+p+1]_0$ by the forward Euler method (3.6), giving

$$|Y_l^k\rangle = \begin{cases} 0, & \text{if } l \in [k]_0; \\ (I+Ah)^{l-k}|b^k\rangle, & \text{if } l \in [m+1]_0 \setminus [k]_0; \\ (I+Ah)^{m-k}|b^k\rangle, & \text{if } l \in [m+p+1]_0 \setminus [m+1]_0. \end{cases} \tag{4.62}$$

Since $\|I+Ah\| \le 1$ by (4.40), we have

$$\begin{aligned} \||Y^k\rangle\|^2 &= \sum_{l=0}^{m+p} \||Y_l^k\rangle\|^2 = \sum_{l=k}^{m} \|I+Ah\|^{l-k}\||b^k\rangle\|^2 + \sum_{l=m+1}^{m+p} \|I+Ah\|^{m-k}\||b^k\rangle\|^2 \\ &\le (m+p+1-k)\||b^k\rangle\|^2 \le (m+p+1)\||b^k\rangle\|^2. \end{aligned} \tag{4.63}$$

Second, for fixed $k \in [m+p+1]_0 \setminus [m+1]_0$, similarly to (4.62), we directly calculate $|Y_l^k\rangle$ using (3.6), giving

$$|Y_l^k\rangle = \begin{cases} 0, & \text{if } l \in [k]_0; \\ |b^k\rangle, & \text{if } l \in [m+p+1]_0 \setminus [k]_0. \end{cases} \tag{4.64}$$

Similarly to (4.63), we have

$$\||Y^k\rangle\|^2 = \sum_{l=0}^{m+p} \||Y_l^k\rangle\|^2 = \sum_{l=k}^{m+p} \||b^k\rangle\|^2 = (m+p+1-k)\||b^k\rangle\|^2 \le (m+p+1)\||b^k\rangle\|^2. \tag{4.65}$$

Combining (4.63) and (4.65), for any $k \in [m+p+1]_0$, we have

$$\||Y^k\rangle\|^2 = \|L^{-1}|b^k\rangle\|^2 \le (m+p+1)\||b^k\rangle\|^2. \tag{4.66}$$

By the definition of $|Y^k\rangle$ in (4.61), (4.66) gives

$$\|L^{-1}\|^2 = \sup_{\||B\rangle\|\le 1} \|L^{-1}|B\rangle\|^2 \le (m+p+1) \sup_{\||b^k\rangle\|\le 1} \|L^{-1}|b^k\rangle\|^2 \le (m+p+1)^2, \tag{4.67}$$

and therefore

$$\|L^{-1}\| \le (m+p+1). \tag{4.68}$$

Finally, combining (4.57) with (4.68), we conclude

$$\kappa = \|L\|\|L^{-1}\| \le 3(m+p+1) \tag{4.69}$$

as claimed. □

4.3 State preparation

We now describe a procedure for preparing the initial state. The initial vector $\hat{y}_{\text{in}}$ is a direct sum over spaces of different dimensions, which is cumbersome to prepare. Instead, we prepare an equivalent state that has a convenient tensor product structure. Specifically, we embed $\hat{y}_{\text{in}}$ into a slightly larger space and prepare the normalized version of

$$\hat{z}_{\text{in}} = [u_{\text{in}} \otimes v_0^{N-1}; u_{\text{in}}^{\otimes 2} \otimes v_0^{N-2}; \ldots; u_{\text{in}}^{\otimes N}], \tag{4.70}$$

where v_0 is some standard vector (for simplicity, we take $v_0 = |0\rangle$). If u_{in} lives in a vector space of dimension n, then $\hat{z}_{\text{in}}$ lives in a space of dimension Nn^N while $\hat{y}_{\text{in}}$ lives in a slightly smaller space of dimension $\Delta = n + n^2 + \cdots + n^N = (n^{N+1}-n)/(n-1)$. Using standard techniques, all the operations we would otherwise apply to $\hat{y}_{\text{in}}$ can be applied instead to $\hat{z}_{\text{in}}$, with the same effect.

Lemma 5. *Assume we are given the value $\|u_{\text{in}}\|$, and let O_x be an oracle that maps $|00\dots0\rangle \in \mathbb{C}^n$ to a normalized quantum state $|u_{\text{in}}\rangle$ proportional to u_{in}. Then the quantum state $|\hat{z}_{\text{in}}\rangle$ proportional to $\hat{z}_{\text{in}} = [u_{\text{in}} \otimes v_0^{N-1}; u_{\text{in}}^{\otimes 2} \otimes v_0^{N-2}; \dots; u_{\text{in}}^{\otimes N}]$ can be prepared with gate complexity $O(\log N)$, using $O(N)$ queries to O_x.*

Proof. Our goal is to prepare the state

$$|\hat{z}_{\text{in}}\rangle = \frac{1}{\sqrt{V}} \sum_j \|u_{\text{in}}\|^j |j\rangle |u_{\text{in}}\rangle^{\otimes j} |0\rangle^{\otimes N-j}, \tag{4.71}$$

where

$$V \coloneqq \sum_{j=1}^{N} \|u_{\text{in}}\|^{2j}. \tag{4.72}$$

This state can be prepared using N queries to the initial state oracle O_x applied in superposition to the intermediate state

$$|\psi_{\text{int}}\rangle \coloneqq \frac{1}{\sqrt{V}} \sum_j \|u_{\text{in}}\|^j |j\rangle \otimes |0\rangle^{\otimes N}. \tag{4.73}$$

To efficiently prepare $|\psi_{\text{int}}\rangle$, notice that

$$|\psi_{\text{int}}\rangle = \frac{1}{\sqrt{V}} \sum_{j_0 j_1 \dots j_k} \prod_{\ell=0}^{k} \|u_{\text{in}}\|^{j_\ell 2^\ell} |j_0 j_1 \dots j_k\rangle \otimes |0\rangle^{\otimes N}, \tag{4.74}$$

where $k \coloneqq \log_2 N$ and $j_0 j_1 \dots j_k$ is the k-bit binary expansion of j. Observe that

$$|\psi_{\text{int}}\rangle = \bigotimes_{\ell=0}^{k} \left(\frac{1}{\sqrt{V_\ell}} \sum_{j_\ell=0}^{1} \|u_{\text{in}}\|^{j_\ell 2^\ell} |j_\ell\rangle \right) \otimes |0\rangle^{\otimes N} \tag{4.75}$$

where

$$V_\ell \coloneqq 1 + \|u_{\text{in}}\|^{2^{\ell+1}}. \tag{4.76}$$

(Notice that $\prod_\ell V_\ell = V$.) Each tensor factor in (4.75) is a qubit that can be produced in constant time. Overall, we prepare these $k = \log N$ qubit states and then apply the state preparation oracle N times, giving the claimed complexity. □

4.4 Measurement success probability

After applying the QLSA to (3.7), we perform a measurement to extract a final state of the desired form. We now consider the probability of this measurement succeeding.

Lemma 6. *Consider an instance of the quantum quadratic ODE problem defined in Problem 1, with the QLSA applied to the linear system (3.7) using the forward Euler method (3.6) with time step (4.23). Suppose the exact solution $u(T)$ and the approximate solution y_1^m defined in (3.7) satisfy*

$$\|u(T) - y_1^m\| \leq \frac{g}{2}, \tag{4.77}$$

where g is defined in (3.11). Then the probability of measuring a state $|y_1^k\rangle$ for $k = [m+p+1]_0 \setminus [m+1]_0$ satisfies

$$P_{\text{measure}} \geq \frac{p+1}{4(m+p+1)Nq^2}, \tag{4.78}$$

where q is also defined in (3.11).

Proof. The idealized quantum state produced by the QLSA applied to (3.7) has the form

$$|Y\rangle = \sum_{k=0}^{m+p} |y^k\rangle|k\rangle = \sum_{k=0}^{m+p}\sum_{j=1}^{N} |y_j^k\rangle|j\rangle|k\rangle \tag{4.79}$$

where the states $|y^k\rangle$ and $|y_j^k\rangle$ for $k \in [m+p+1]_0$ and $j \in [N]$ are subnormalized to ensure $\||Y\rangle\| = 1$.

We decompose the state $|Y\rangle$ as

$$|Y\rangle = |Y_{\text{bad}}\rangle + |Y_{\text{good}}\rangle, \tag{4.80}$$

where

$$\begin{aligned} |Y_{\text{bad}}\rangle &\coloneqq \sum_{k=0}^{m-1}\sum_{j=1}^{N} |y_j^k\rangle|j\rangle|k\rangle + \sum_{k=m}^{m+p}\sum_{j=2}^{N} |y_j^k\rangle|j\rangle|k\rangle, \\ |Y_{\text{good}}\rangle &\coloneqq \sum_{k=m}^{m+p} |y_1^k\rangle|1\rangle|k\rangle. \end{aligned} \tag{4.81}$$

Note that $|y_1^k\rangle = |y_1^m\rangle$ for all $k \in \{m, m+1, \ldots, m+p\}$. We lower bound

$$P_{\text{measure}} \coloneqq \frac{\||Y_{\text{good}}\rangle\|^2}{\||Y\rangle\|^2} = \frac{(p+1)\||y_1^m\rangle\|^2}{\||Y\rangle\|^2} \tag{4.82}$$

by lower bounding the terms of the product

$$\frac{\||y_1^m\rangle\|^2}{\||Y\rangle\|^2} = \frac{\||y_1^m\rangle\|^2}{\||y_1^0\rangle\|^2} \cdot \frac{\||y_1^0\rangle\|^2}{\||y^0\rangle\|^2} \cdot \frac{\||y^0\rangle\|^2}{\||Y\rangle\|^2}. \tag{4.83}$$

First, since $y_1^0 = (y_{\text{in}})_1 = u_{\text{in}}$, using (3.11) and (4.77), we have

$$\frac{\||y_1^m\rangle\|}{\||y_1^0\rangle\|} = \frac{\|y_1^m\|}{\|u_{\text{in}}\|} \geq \frac{\|u(T)\| - \|u(T) - y_1^m\|}{\|u_{\text{in}}\|} = \frac{g - \|u(T) - y_1^m\|}{\|u_{\text{in}}\|} \geq \frac{g}{2\|u_{\text{in}}\|} = \frac{1}{2q}. \tag{4.84}$$

Second, for $|y^0\rangle = |y_{\text{in}}\rangle$ with $(y_{\text{in}})_j = u_{\text{in}}^{\otimes j}$, we have $\||y_j^0\rangle\|^2 = \|u_{\text{in}}\|^{2j}$. Therefore

$$\||y^0\rangle\|^2 = \sum_{j=1}^{N} \|u_{\text{in}}\|^{2j} \leq N\|u_{\text{in}}\|^2 \tag{4.85}$$

(using $\|u_{\text{in}}\| \leq 1$ by (2.4)), so

$$\frac{\||y_1^0\rangle\|^2}{\||y^0\rangle\|^2} \geq \frac{1}{N}. \tag{4.86}$$

Third, the forward Euler method (3.6) gives

$$|y^k\rangle = \begin{cases} (I + Ah)^k|y^0\rangle, & \text{if } k \in [m+1]_0; \\ (I + Ah)^m|y^0\rangle, & \text{if } k \in [m+p+1]_0 \setminus [m+1]_0, \end{cases} \tag{4.87}$$

and therefore

$$\||y^k\rangle\|^2 \leq \|I + Ah\|^{2m}\||y^0\rangle\|^2 \tag{4.88}$$

for all $k \in [m+p+1]_0$. By (4.40), we have $\|I + Ah\| \le 1$, so $\||y^k\rangle\|^2 \le \||y^0\rangle\|^2$, and therefore

$$\frac{\||y^0\rangle\|^2}{\||Y\rangle\|^2} = \frac{\||y^0\rangle\|^2}{\sum_{k=0}^{m+p} \||y^k\rangle\|^2} \ge \frac{1}{m+p+1}. \tag{4.89}$$

Finally, using (4.84), (4.86), and (4.89) in (4.83) and (4.82), we have

$$P_{\text{measure}} \ge \frac{p+1}{4(m+p+1)Nq^2} \tag{4.90}$$

as claimed. □

Choosing $m = p$, we have $P_{\text{measure}} = \Omega(1/Nq^2)$. Using amplitude amplification, $O(\sqrt{N}q)$ iterations suffice to succeed with constant probability.

4.5 Proof of Theorem 1

Proof. We first present the quantum Carleman linearization (QCL) algorithm and then analyze its complexity.

The QCL algorithm. We start by rescaling the system to satisfy (2.3) and (2.4). Given a quadratic ODE (2.1) satisfying $\mathrm{R} < 1$ (where R is defined in (2.2)), we define a scaling factor $\gamma > 0$, and rescale u to

$$\overline{u} \coloneqq \gamma u. \tag{4.91}$$

Replacing u by $\overline{u}$ in (2.1), we have

$$\begin{aligned} \frac{\mathrm{d}\overline{u}}{\mathrm{d}t} &= F_1\overline{u} + \frac{1}{\gamma}F_2\overline{u}^{\otimes 2}, \\ \overline{u}(0) &= \overline{u}_{\text{in}} \coloneqq \gamma u_{\text{in}}. \end{aligned} \tag{4.92}$$

We let $\overline{F}_1 \coloneqq F_1$ and $\overline{F}_2 \coloneqq \frac{1}{\gamma}F_2$ so that

$$\begin{aligned} \frac{\mathrm{d}\overline{u}}{\mathrm{d}t} &= \overline{F}_1\overline{u} + \overline{F}_2\overline{u}^{\otimes 2}, \\ \overline{u}(0) &= \overline{u}_{\text{in}}. \end{aligned} \tag{4.93}$$

Note that R is invariant under this rescaling, so $\mathrm{R} < 1$ still holds for the rescaled equation.

Concretely, we take[1]

$$\gamma = \frac{1}{\|u_{\text{in}}\|}. \tag{4.94}$$

After rescaling, the new quadratic ODE satisfies $\|\overline{u}_{\text{in}}\| = \gamma\|u_{\text{in}}\| = 1$, so (2.4) holds. Furthermore, the assumption $\mathrm{R} < 1$ implies $\|\overline{F_2}\| = \|u_{\text{in}}\|\|F_2\| < |\operatorname{Re}(\overline{\lambda_1})|$, so (2.3) holds for the rescaled problem.

Having performed this rescaling, we henceforth assume that (2.3) and (2.4) are satisfied. We then introduce the choice of parameters as follows. Given g and an error bound $\epsilon \le 1$, we define

$$\delta \coloneqq \frac{g\epsilon}{1+\epsilon} \le \frac{g}{2}. \tag{4.95}$$

Given $\|u_{\text{in}}\|$, $\|F_2\|$, and $\operatorname{Re}(\lambda_1) < 0$, we choose

$$N = \left\lceil \frac{\log(|\operatorname{Re}(\lambda_1)|/\|u_{\text{in}}\|\|F_2\|)}{\log(2\|u_{\text{in}}\|/\delta)} \right\rceil = \left\lceil \frac{\log(1/\mathrm{R})}{\log(2\|u_{\text{in}}\|/\delta)} \right\rceil \tag{4.96}$$

[1] In fact, one can show that any $\gamma \in \left(\frac{\|F_2\|}{|\operatorname{Re}(\lambda_1)|}, \frac{1}{\|u_{\text{in}}\|}\right]$ suffices to satisfy (2.3) and (2.4).

(recalling the definition of R in (2.2)). Since $\|u_{\text{in}}\|/\delta > 1$ by (4.95) and $g < \|u_{\text{in}}\|$, Lemma 2 gives

$$\|u(T) - \hat{y}_1(T)\| \le \|u_{\text{in}}\| \, \mathrm{R}^N \le \frac{\delta}{2}. \tag{4.97}$$

Now we discuss the choice of h. On the one hand, h must satisfy (4.23) to satisfy the conditions of Lemma 3 and Lemma 4. On the other hand, Lemma 3 gives the upper bound

$$\|\hat{y}_1(T) - y_1^m\| \le N^{2.5} T h (\|F_1\| + \|F_2\|)^2 \le \frac{g\epsilon}{4} \le \frac{g\epsilon}{2(1+\epsilon)} = \frac{\delta}{2}. \tag{4.98}$$

Thus, we choose

$$h \le \min\left\{ \frac{g\epsilon}{4N^{2.5}T(\|F_1\| + \|F_2\|)^2}, \frac{1}{N\|F_1\|}, \frac{2(|\operatorname{Re}(\lambda_1)| - \|F_2\|)}{N(|\operatorname{Re}(\lambda_1)|^2 - \|F_2\|^2 + \|F_1\|^2)} \right\} \tag{4.99}$$

to satisfy (4.23) and (4.98).

Combining (4.97) with (4.98), we have

$$\|u(T) - y_1^m\| \le \|u(T) - \hat{y}_1(T)\| + \|\hat{y}_1(T) - y_1^m\| \le \delta. \tag{4.100}$$

Thus, (4.77) holds since $\delta \le g/2$. Using

$$\left\| \frac{u(T)}{\|u(T)\|} - \frac{y_1^m}{\|y_1^m\|} \right\| \le \frac{\|u(T) - y_1^m\|}{\min\{\|u(T)\|, \|y_1^m\|\}} \le \frac{\|u(T) - y_1^m\|}{g - \|u(T) - y_1^m\|} \tag{4.101}$$

and (4.100), we obtain

$$\left\| \frac{u(T)}{\|u(T)\|} - \frac{y_1^m}{\|y_1^m\|} \right\| \le \frac{\delta}{g - \delta} = \epsilon, \tag{4.102}$$

i.e., the normalized output state is ϵ-close to $\frac{u(T)}{\|u(T)\|}$.

We follow the procedure in Lemma 5 to prepare the initial state $|\hat{y}_{\text{in}}\rangle$. We apply the QLSA [18] to the linear system (3.7) with $m = p = \lceil T/h \rceil$, giving a solution $|Y\rangle$. We then perform a measurement to obtain a normalized state of $|y_j^k\rangle$ for some $k \in [m+p+1]_0$ and $j \in [N]$. By Lemma 6, the probability of obtaining a state $|y_1^k\rangle$ for some $k \in [m+p+1]_0 \setminus [m+1]_0$, giving the normalized vector $y_1^m/\|y_1^m\|$, is

$$P_{\text{measure}} \ge \frac{p+1}{4(m+p+1)Nq^2} \ge \frac{1}{8Nq^2}. \tag{4.103}$$

By amplitude amplification, we can achieve success probability $\Omega(1)$ with $O(\sqrt{N}q)$ repetitions of the above procedure.

Analysis of the complexity. By Lemma 5, the initial state $|\hat{y}_{\text{in}}\rangle$ can be prepared with $O(N)$ queries to O_x, with gate complexity larger by a $\operatorname{poly}(\log n)$ factor. The matrix L is a $(m+p+1)\Delta \times (m+p+1)\Delta$ matrix with $O(Ns)$ nonzero entries in any row or column. By Lemma 4 and

our choice of parameters, the condition number of L is at most

$$\begin{aligned}
&3(m+p+1)\\
&= O\left(\frac{N^{2.5}T^2(\|F_1\|+\|F_2\|)^2}{\delta} + NT\|F_1\| + \frac{NT(|\operatorname{Re}(\lambda_1)|^2 - \|F_2\|^2 + \|F_1\|^2)}{2(|\operatorname{Re}(\lambda_1)| - \|F_2\|)}\right)\\
&= O\left(\frac{N^{2.5}T^2(\|F_1\|+\|F_2\|)^2}{\delta} + NT\|F_1\| + \frac{NT}{2}(|\operatorname{Re}(\lambda_1)| + \|F_2\|) + \frac{NT\|F_1\|^2}{2(|\operatorname{Re}(\lambda_1)| - \|F_2\|)}\right)\\
&= O\left(\frac{N^{2.5}T^2(\|F_1\|+\|F_2\|)^2}{g\epsilon} + \frac{\mathrm{R}}{1-\mathrm{R}}\frac{NT\|F_1\|^2}{\|F_2\|}\right)\\
&= O\left(\frac{N^{2.5}T^2(\|F_1\|+\|F_2\|)^2}{(1-\mathrm{R})\|F_2\|g\epsilon}\right).
\end{aligned} \tag{4.104}$$

Here we use[2] $\mathrm{R}\,|\operatorname{Re}(\lambda_1)| = \|F_2\| < |\operatorname{Re}(\lambda_1)| \le \|F_1\|$ and $1/(1-\mathrm{R}) > 1$. Consequently, by Theorem 5 of [18], the QLSA produces the state $|Y\rangle$ with

$$\begin{aligned}
&\frac{N^{3.5}sT^2(\|F_1\|+\|F_2\|)^2}{(1-\mathrm{R})\|F_2\|g\epsilon} \cdot \operatorname{poly}\left(\log\frac{NsT\|F_1\|\|F_2\|}{(1-\mathrm{R})g\epsilon}\right)\\
&= \frac{sT^2(\|F_1\|+\|F_2\|)^2}{(1-\mathrm{R})\|F_2\|g\epsilon} \cdot \operatorname{poly}\left(\log\frac{sT\|F_1\|\|F_2\|\log(1/R)}{(1-\mathrm{R})g\epsilon}\right)
\end{aligned} \tag{4.105}$$

queries to the oracles O_{F_1} and O_{F_2}. Using $O(\sqrt{N}q)$ steps of amplitude amplification to achieve success probability $\Omega(1)$, the overall query complexity of our algorithm is

$$\begin{aligned}
&\frac{N^4sT^2q(\|F_1\|+\|F_2\|)^2}{(1-\mathrm{R})\|F_2\|g\epsilon} \cdot \operatorname{poly}\left(\log\frac{NsT\|F_1\|\|F_2\|}{(1-\mathrm{R})g\epsilon}\right)\\
&= \frac{sT^2q(\|F_1\|+\|F_2\|)^2}{(1-\mathrm{R})\|F_2\|g\epsilon} \cdot \operatorname{poly}\left(\log\frac{sT\|F_1\|\|F_2\|\log(1/R)}{(1-\mathrm{R})g\epsilon}\right)
\end{aligned} \tag{4.106}$$

and the gate complexity is larger by a factor of $\operatorname{poly}(\log(nsT\|F_1\|\|F_2\|\log(1/R)/(1-\mathrm{R})g\epsilon))$.

If the eigenvalues λ_j of F_1 are all real, by (4.24), the condition number of L is at most

$$3(m+p+1) = O\left(\frac{N^{2.5}T^2(\|F_1\|+\|F_2\|)^2}{\delta} + NT\|F_1\|\right) = O\left(\frac{N^{2.5}T^2(\|F_1\|+\|F_2\|)^2}{g\epsilon}\right). \tag{4.107}$$

Similarly, the QLSA produces the state $|Y\rangle$ with

$$\frac{sT^2(\|F_1\|+\|F_2\|)^2}{g\epsilon} \cdot \operatorname{poly}\left(\log\frac{sT\|F_1\|\|F_2\|\log(1/R)}{g\epsilon}\right) \tag{4.108}$$

queries to the oracles O_{F_1} and O_{F_2}. Using amplitude amplification to achieve success probability $\Omega(1)$, the overall query complexity of the algorithm is

$$\frac{sT^2q(\|F_1\|+\|F_2\|)^2}{g\epsilon} \cdot \operatorname{poly}\left(\log\frac{sT\|F_1\|\|F_2\|\log(1/R)}{g\epsilon}\right) \tag{4.109}$$

and the gate complexity is larger by a factor of $\operatorname{poly}(\log(nsT\|F_1\|\|F_2\|\log(1/R)/g\epsilon))$ as claimed. □

[2] If we choose γ differently from (4.94), which is allowed by footnote 1, then using $\|F_2\|\|u_{\mathrm{in}}\| = \mathrm{R}\,|\operatorname{Re}(\lambda_1)|$, it suffices to replace R by $\mathrm{R}/\|u_{\mathrm{in}}\|$ in (4.104) and subsequent expressions.

5 Lower bound

In this section, we establish a limitation on the ability of quantum computers to solve the quadratic ODE problem when the nonlinearity is sufficiently strong. We quantify the strength of the nonlinearity in terms of the quantity R defined in (2.2). Whereas there is an efficient quantum algorithm for $\mathrm{R} < 1$ (as shown in Theorem 1), we show here that the problem is intractable for $\mathrm{R} \geq \sqrt{2}$.

Theorem 2. *Assume* $\mathrm{R} \geq \sqrt{2}$. *Then there is an instance of the quantum quadratic ODE problem defined in Problem 1 such that any quantum algorithm for producing a quantum state approximating* $u(T)/\|u(T)\|$ *with bounded error must have worst-case time complexity exponential in* T.

We establish this result by showing how the nonlinear dynamics can be used to distinguish nonorthogonal quantum states, and thereby solve unstructured search in a time that violates the quantum search lower bound [7]. Note that since our algorithm only approximates the quantum state corresponding to the solution, we must lower bound the query complexity of *approximating* the solution of a quadratic ODE.

5.1 Unstructured search and state discrimination

In the unstructured search problem, we aim to search the set $[n]$ for a marked item, given the ability to query whether any given $x \in [n]$ is marked. In the decision version of the problem, our goal is to determine whether there are any marked items. It is well known that the classical query complexity of unstructured search is $\Theta(n)$, while the quantum query complexity is $\Theta(\sqrt{n})$ [7, 32].

Previous work on the computational power of nonlinear quantum mechanics shows that the ability to distinguish non-orthogonal states can be applied to solve unstructured search (and other hard computational problems) [1, 2, 21]. In particular, this connection shows the following.

Lemma 7 ([21])**.** *Let* $|\psi\rangle, |\phi\rangle$ *be states of a qubit with* $|\langle\psi|\phi\rangle| = 1 - \epsilon$. *Suppose we are either given a black box that prepares* $|\psi\rangle$ *or a black box that prepares* $|\phi\rangle$, *where a call to the black box takes time* 1. *Then any bounded-error protocol for determining whether the state is* $|\psi\rangle$ *or* $|\phi\rangle$ *must take time* $\Omega(1/\epsilon^{1/4})$.

Proof. The claim follows from the construction in Section 3.1 of [21], taking $t_1 = \pi$ to consider the discrete quantum query model. Given a black box for the unstructured search problem, the Hadamard test circuit can be used to create a single-qubit state that is either $|\psi\rangle = |0\rangle$ (if there are no marked items) or a state $|\phi\rangle$ that has overlap $1 - \epsilon$ with $|0\rangle$, where $\epsilon = \frac{1}{2n^2} + O(1/n^4)$. This preparation procedure succeeds with probability $1 - O(1/n)$, so it has negligible probability of ever failing in an algorithm that runs in time $o(n)$. Since the bounded-error quantum query complexity of unstructured search is $\Omega(\sqrt{n})$ [7], any bounded-error algorithm for distinguishing the states must take time $\Omega(\sqrt{n}) = \Omega(1/\epsilon^{1/4})$. □

5.2 State discrimination with nonlinear dynamics

Lemma 7 can be used to establish limitations on the ability of quantum computers to simulate nonlinear dynamics, since nonlinear dynamics can be used to distinguish nonorthogonal states. Whereas previous work considers models of nonlinear quantum dynamics (such as the Weinberg model [1, 2] and the Gross-Pitaevskii equation [21]), here we aim to show the difficulty of efficiently simulating more general nonlinear ODEs—in particular, quadratic ODEs with dissipation—using quantum algorithms.

Lemma 8. *There exists an instance of the quantum quadratic ODE problem as defined in Problem 1 with* $\mathrm{R} \geq \sqrt{2}$*, and two states of a qubit with overlap* $1-\epsilon$ *(for any sufficiently small* $\epsilon > 0$*) as possible initial conditions, such that the two final states after evolution time* $T = O(\log(1/\epsilon))$ *have an overlap no larger than* $3/\sqrt{10}$*.*

Proof. Consider a 2-dimensional system of the form

$$\begin{aligned} \frac{\mathrm{d}u_1}{\mathrm{d}t} &= -u_1 + ru_1^2, \\ \frac{\mathrm{d}u_2}{\mathrm{d}t} &= -u_2 + ru_2^2, \end{aligned} \tag{5.1}$$

for some $r > 0$, with an initial condition $u(0) = [u_1(0); u_2(0)] = u_{\mathrm{in}}$ satisfying $\|u_{\mathrm{in}}\| = 1$. According to the definition of R in (2.2), we have $\mathrm{R} = r$, so henceforth we write this parameter as R. The analytic solution of (5.1) is

$$\begin{aligned} u_1(t) &= \frac{1}{\mathrm{R} - e^t(\mathrm{R} - 1/u_1(0))}, \\ u_2(t) &= \frac{1}{\mathrm{R} - e^t(\mathrm{R} - 1/u_2(0))}. \end{aligned} \tag{5.2}$$

When $u_2(0) > 1/\mathrm{R}$, $u_2(t)$ is finite within the domain

$$0 \leq t < t^* := \log\left(\frac{\mathrm{R}}{\mathrm{R} - 1/u_2(0)}\right); \tag{5.3}$$

when $u_2(0) = 1/\mathrm{R}$, we have $u_2(t) = 1/\mathrm{R}$ for all t; and when $u_2(0) < 1/\mathrm{R}$, $u_2(t)$ goes to 0 as $t \to \infty$. The behavior of $u_1(t)$ depends similarly on $u_1(0)$.

Without loss of generality, we assume $u_1(0) \leq u_2(0)$. For $u_2(0) \geq u_1(0) > 1/\mathrm{R}$, both $u_1(t)$ and $u_2(t)$ are finite within the domain (5.3), which we consider as the domain of $u(t)$.

Now we consider 1-qubit states that provide inputs to (5.1). Given a sufficiently small $\epsilon > 0$, we first define $\theta \in (0, \pi/4)$ by

$$2\sin^2\frac{\theta}{2} = \epsilon. \tag{5.4}$$

We then construct two 1-qubit states with overlap $1-\epsilon$, namely

$$|\phi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{5.5}$$

and

$$|\psi(0)\rangle = \cos\left(\theta + \frac{\pi}{4}\right)|0\rangle + \sin\left(\theta + \frac{\pi}{4}\right)|1\rangle. \tag{5.6}$$

Then the overlap between the two initial states is

$$\langle\phi(0)|\psi(0)\rangle = \cos\theta = 1 - \epsilon. \tag{5.7}$$

The initial overlap (5.7) is larger than the target overlap $3/\sqrt{10}$ in Lemma 8 provided $\epsilon < 1 - 3/\sqrt{10}$. For simplicity, we denote

$$\begin{aligned} v_0 &:= \cos\left(\theta + \frac{\pi}{4}\right), \\ w_0 &:= \sin\left(\theta + \frac{\pi}{4}\right), \end{aligned} \tag{5.8}$$

and let $v(t)$ and $w(t)$ denote solutions of (5.1) with initial conditions $v(0) = v_0$ and $w(0) = w_0$, respectively. Since $w_0 > 1/\mathrm{R}$, we see that $w(t)$ increases with t, satisfying

$$\frac{1}{\mathrm{R}} \leq \frac{1}{\sqrt{2}} < w_0 < w(t), \tag{5.9}$$

and

$$v(t) < w(t) \tag{5.10}$$

for any time $0 < t < t^*$, whatever the behavior of $v(t)$.

We now study the outputs of our problem. For the state $|\phi(0)\rangle$, the initial condition for (5.1) is $[1/\sqrt{2}; 1/\sqrt{2}]$. Thus, the output for any $t \geq 0$ is

$$|\phi(t)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle). \tag{5.11}$$

For the state $|\psi(0)\rangle$, the initial condition for (5.1) is $[v_0; w_0]$. We now discuss how to select a terminal time T to give a useful output state $|\psi(T)\rangle$. For simplicity, we denote the ratio of $w(t)$ and $v(t)$ by

$$K(t) \coloneqq \frac{w(t)}{v(t)}. \tag{5.12}$$

Noticing that $w(t)$ goes to infinity as t approaches t^*, while $v(t)$ remains finite within (5.3), there exists a terminal time T such that[3]

$$K(T) \geq 2. \tag{5.13}$$

The normalized output state at this time T is

$$|\psi(T)\rangle = \frac{1}{\sqrt{K(T)^2 + 1}}(|0\rangle + K(T)|1\rangle). \tag{5.14}$$

Combining (5.11) with (5.14), the overlap of $|\phi(T)\rangle$ and $|\psi(T)\rangle$ is

$$\langle\phi(T)|\psi(T)\rangle = \frac{K(T) + 1}{\sqrt{2K(T)^2 + 2}} \leq \frac{3}{\sqrt{10}} \tag{5.15}$$

using (5.13).

Finally, we estimate the evolution time T, which is implicitly defined by (5.13). We can upper bound its value by t^*. According to (5.3), we have

$$T < t^* = \log\left(\frac{\mathrm{R}}{\mathrm{R} - \frac{1}{w_0}}\right) < \log\left(\frac{\sqrt{2}}{\sqrt{2} - \frac{1}{w_0}}\right) \tag{5.16}$$

since the function $\log(x/(x-c))$ decreases monotonically with x for $x > c > 0$. Using (5.7) and (5.8) to rewrite this expression in terms of ϵ, we have

$$T < t^* < \log\left(\frac{\sqrt{2}}{\sqrt{2} - \frac{1}{\sin(\theta + \frac{\pi}{4})}}\right) = \log\left(\frac{\sin\theta + \cos\theta}{\sin\theta + \cos\theta - 1}\right) = \log\left(\frac{\sqrt{2\epsilon - \epsilon^2} + 1 - \epsilon}{\sqrt{2\epsilon - \epsilon^2} - \epsilon}\right). \tag{5.17}$$

For ϵ sufficiently small, this implies $T = O(\log(1/\epsilon))$ as claimed. □

[3]More concretely, we take $v_{\max} = \max\{v_0, v(t^*)\}$ that upper bounds $v(t)$ on the domain $[0, t^*)$, in which $v(t^*)$ is a finite value since $v_0 < w_0$. Then there exists a terminal time T such that $w(T) = 2v_{\max}$, and hence $K(T) = w(T)/v(T) \geq 2$.

5.3 Proof of Theorem 2

We now establish our main lower bound result.

Proof. As introduced in the proof of Lemma 8, consider the quadratic ODE (5.1); the two initial states of a qubit $|\phi(0)\rangle$ and $|\psi(0)\rangle$ defined in (5.5) and (5.6), respectively; and the terminal time T defined in (5.13).

Suppose we have a quantum algorithm that, given a black box to prepare a state that is either $|\phi(0)\rangle$ or $|\psi(0)\rangle$, can produce quantum states $|\phi'(T)\rangle$ or $|\psi'(T)\rangle$ that are within distance δ of $|\phi(T)\rangle$ and $|\psi(T)\rangle$, respectively. Since by Lemma 8, $|\phi(T)\rangle$ and $|\psi(T)\rangle$ have constant overlap, the overlap between $|\phi'(T)\rangle$ and $|\psi'(T)\rangle$ is also constant for sufficiently small δ. More precisely, we have

$$\langle\phi(T)|\psi(T)\rangle \leq \frac{3}{\sqrt{10}} \tag{5.18}$$

by (5.15), which implies

$$\| |\phi(T)\rangle - |\psi(T)\rangle \| \geq \sqrt{2\left(1 - \frac{3}{\sqrt{10}}\right)} > 0.32. \tag{5.19}$$

We also have

$$\| \, |\phi(T)\rangle - |\phi'(T)\rangle \, \| \leq \delta, \tag{5.20}$$

and similarly for $\psi(T)$. These three inequalities give us

$$\begin{aligned} \| \, |\phi'(T)\rangle - |\psi'(T)\rangle \, \| &= \| (|\phi(T)\rangle - |\psi(T)\rangle) - (|\phi(T)\rangle - |\phi'(T)\rangle) - (|\psi'(T)\rangle - |\psi(T)\rangle) \, \| \\ &\geq \| (|\phi(T)\rangle - |\psi(T)\rangle) \, \| - \| (|\phi(T)\rangle - |\phi'(T)\rangle) \, \| - \| (|\psi'(T)\rangle - |\psi(T)\rangle) \, \| \\ &> 0.32 - 2\delta, \end{aligned} \tag{5.21}$$

which is at least a constant for (say) $\delta < 0.15$.

Lemma 7 therefore shows that preparing the states $|\phi'(T)\rangle$ and $|\psi'(T)\rangle$ requires time $\Omega(1/\epsilon^{1/4})$, as these states can be used to distinguish the two possibilities with bounded error. By Lemma 8, this time is $2^{\Omega(T)}$. This shows that we need at least exponential simulation time to approximate the solution of arbitrary quadratic ODEs to within sufficiently small bounded error when $\mathrm{R} \geq \sqrt{2}$. □

Note that exponential time is achievable since our QCL algorithm can solve the problem by taking N to be exponential in T, where N is the truncation level of Carleman linearization. (The algorithm of Leyton and Osborne also solves quadratic differential equations with complexity exponential in T, but requires the additional assumptions that the quadratic polynomial is measure-preserving and Lipschitz continuous [39].)

6 Applications

Our QCL algorithm only applies to problems with linear dissipation, i.e., $\mathrm{Re}(\lambda_1) < 0$, and a weak nonlinearity in the sense that $\|u_{\mathrm{in}}\|\|F_2\| < |\mathrm{Re}(\lambda_1)|$ (i.e., $\mathrm{R} < 1$). Thus it is natural to ask which problems of interest satisfy these conditions.

Perhaps the simplest example is based on the logistic differential equation $\frac{\mathrm{d}u}{\mathrm{d}t} = ru(1 - \frac{u}{K})$ for $u(t) \in \mathbb{R}$, which is a simple model used for various growth scenarios [48]. The parameter $r > 0$ controls the initial rate of exponential growth, and $K > 0$ is the asymptotic equilibrium. If we move our frame of reference to the asymptotic equilibrium by considering $v \coloneqq K - u$, then we obtain

$\frac{\mathrm{d}v}{\mathrm{d}t} = -rv + \frac{r}{K}v^2$, where the requirements of our algorithm are clearly satisfied if $0 < v(0) < K$. However, this equation has a closed-form solution and is only one-dimensional, so it does not provide a practical application of our quantum algorithm.

To find an application that can benefit from quantum speedup, we must increase the dimension of the system. While uncoupled logistic differential equations still have a closed-form solution, this is no longer the case if we introduce nonlinear couplings among the variables. Such equations can have the form

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = -r_i u_i + \sum_{j=1}^{n} c_{ij} u_i u_j \tag{6.1}$$

for $i \in [n]$, where $r_i > 0$ are decay rates and $c_{ij} \in \mathbb{R}$ are pairwise nonlinear couplings. Equations of this type can be used to model interacting populations of predators and prey [48], the spread of infection during a pandemic [11], or the dynamics of chemical reactions [17, 43]. Note that the $-r_i u_i$ term models exponential decay, which must be present for our algorithm to apply. In the case of predators and prey, this effect could be caused by hunting; for a pandemic, it could be caused by an irreversible sterilizing effect; and in the case of chemical reactions, it could be due to a neutralizing reactant. Furthermore, to satisfy the requirement $\mathrm{R} < 1$, we require that the initial condition $u_i(0)$ and/or coupling coefficients c_{ij} are sufficiently small compared to the linear decay rate r_i. Whether this is satisfied depends on the details of the problem.

Other examples of high-dimensional ODEs arise from the discretization of certain PDEs. Consider for example equations for $\mathbf{u}(\mathbf{r}, t)$ of the type

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \beta \mathbf{u} = \nu \nabla^2 \mathbf{u}. \tag{6.2}$$

This equation can be cast in the form of (6.1) by the Fourier transform, but can also be cast directly in the form (2.1) using standard discretizations of space and time. Equations of the form (6.2) can represent Navier–Stokes-type equations, which are ubiquitous in fluid mechanics [38], and related models such as those studied in [36, 45, 47] to describe the formation of large-scale structure in the universe. Similar equations also appear in models of magnetohydrodynamics [e.g., 26] or the motion of free particles that stick to each other upon collision [12]. In the inviscid case, $\nu = 0$, the resulting Euler-type equations with linear damping are also of interest, both for modeling micromechanical devices [5] and for their intimate connection with viscous models [24].

As a specific example, consider the one-dimensional viscous Burgers equation

$$\partial_t u + u \partial_x u = \nu \partial_x^2 u, \tag{6.3}$$

which is the one-dimensional case of equation (6.2) with $\beta = 0$. Equation (6.3) is often used as a simple model of turbulence [14]. For concreteness, let the initial condition be $u(x, 0) = U_0 \sin(2\pi x / L_0)$ on the domain $x \in [-L_0/2, L_0/2]$ and use Dirichlet boundary conditions $u(-L_0/2, 0) = u(L_0/2, 0) = 0$. To solve this equation using the Carleman method, we discretize the spatial domain into n points and use central differences for the derivatives to get

$$\partial_t u_i = \nu \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} - \frac{u_{i+1}^2 - u_{i-1}^2}{4\Delta x} \tag{6.4}$$

with $\Delta x = L_0/(n-1)$. This equation is of the form (2.1) and can thus generate the Carleman system (3.2). The resulting linear ODE can then be integrated using the forward Euler method, as shown in Figure 1. In this example, the viscosity ν is defined such that the Reynolds number $\mathrm{Re} \coloneqq U_0 L_0 / \nu = 20$, and $n = 16$ spatial discretization points were sufficient to resolve the solution. The figure compares the Carleman solution with solutions provided by two other methods: a direct

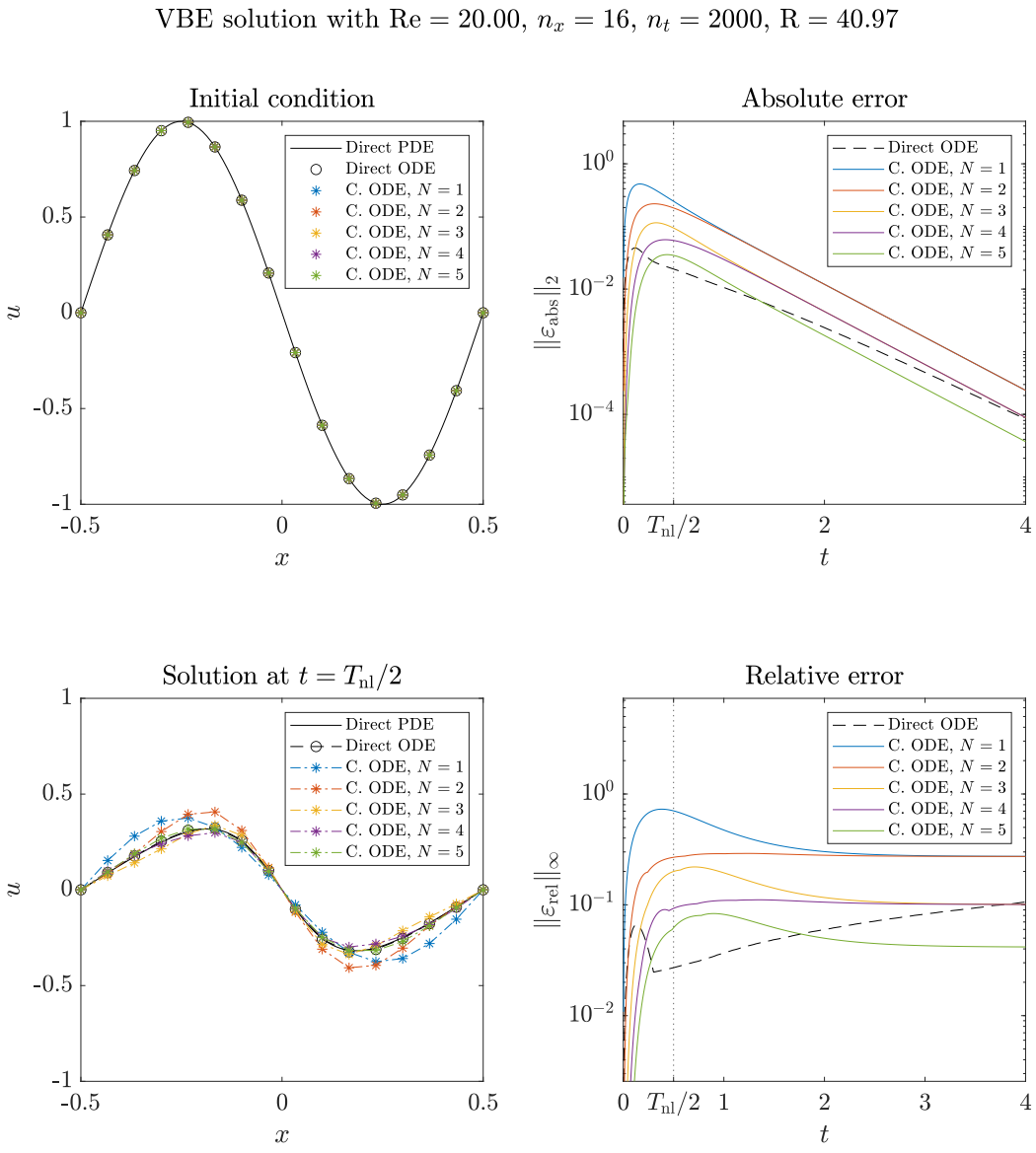

Figure 1: Integration of the viscous Burgers equation using Carleman linearization on a classical computer. The viscosity was set so that the Reynolds number $\mathrm{Re} = U_0 L_0/\nu = 20$. The parameters $n_x = 16$ and $n_t = 2000$ are the number of spatial and temporal discretization intervals, respectively. The corresponding Carleman convergence parameter is $\mathrm{R} = 40.97$. Left: Initial condition and solution plotted at half the nonlinear time $\frac{1}{2}T_{\mathrm{nl}} = \frac{L_0}{2U_0} = \frac{1}{2}$. Right: l_2 norm of the absolute error and l_∞ norm of the relative error, between the Carleman solution (*C. ODE*) and the direct forward Euler solution (*Direct ODE*). The errors between the direct forward Euler solution and the high-resolution `pdepe` solution (*Direct PDE*) are included for reference.

application of the forward Euler method to equation (6.4) without Carleman linearization, and an application of MATLAB's inbuilt PDE solver `pdepe` to the continuous PDE (6.3). By inserting the matrices F_1 and F_2 corresponding to equation (6.4) into the definition of R (2.2), we find that the parameters used in this example result in $\mathrm{R} \approx 41$. Even though this does not satisfy the requirement $\mathrm{R} < 1$ of the QCL algorithm, we see from the absolute error plot in Figure 1 that the maximum absolute error over time decreases uniformly on a logarithmic scale as the truncation level N is incremented (in this example, the maximum Carleman truncation level considered is $N = 5$). Remarkably, this suggests that in this example, the error of the classical Carleman method converges exponentially with N, even though $\mathrm{R} > 1$.

7 Discussion

In this paper we have presented a quantum Carleman linearization (QCL) algorithm for a class of quadratic nonlinear differential equations. Compared to the previous approach of [39], our algorithm improves the complexity from an exponential dependence on T to a nearly quadratic dependence, under the condition $\mathrm{R} < 1$ as defined in (2.2). Qualitatively, this means that the system must be dissipative and that the nonlinear effects must be small relative to the linear effects. We have also provided numerical results suggesting the classical Carleman method may work on certain PDEs that do not strictly satisfy the assumption $\mathrm{R} < 1$. Furthermore, we established a lower bound showing that for general quadratic differential equations with $\mathrm{R} \geq \sqrt{2}$, quantum algorithms must have worst-case complexity exponential in T. We also discussed several potential applications arising in biology and fluid and plasma dynamics.

It is natural to ask whether the result of Theorem 1 can be achieved with a classical algorithm, i.e., whether the assumption $\mathrm{R} < 1$ makes differential equations classically tractable. Clearly a naive integration of the truncated Carleman system (3.2) is not efficient on a classical computer since the system size is $\Theta(n^N)$. But furthermore, it is unlikely that *any* classical algorithm for this problem can run in time polynomial in n. If we consider Problem 1 with dissipation that is small compared to the total evolution time, but let the nonlinearity be even smaller such that $\mathrm{R} < 1$, then in the asymptotic limit we have a linear differential equation with no dissipation. Hence any classical algorithm that could solve Problem 1 could also solve non-dissipative linear differential equations, which is a BQP-hard problem even when the dynamics are unitary [29]. In other words, an efficient classical algorithm for this problem would imply efficient classical algorithms for any problem that can be solved efficiently by a quantum computer, which is considered unlikely.

Our upper and lower bounds leave a gap in the range $1 \leq \mathrm{R} < \sqrt{2}$, for which we do not know the complexity of the quantum quadratic ODE problem. We hope that future work will close this gap and determine for which R the problem can be solved efficiently by quantum computers in the worst case.

Furthermore, the complexity of our algorithm has nearly quadratic dependence on T, namely $T^2 \operatorname{poly}(\log T)$. It is unknown whether the complexity for quadratic ODEs must be at least linear or quadratic in T. Note that a sublinear complexity is unlikely because of the no-fast-forwarding theorem [9].

The complexity of our algorithm depends on the parameter q defined in (3.11), which characterizes the decay of the final solution relative to the initial condition. This restricts the utility of our result, since we are required to select a proper initial condition and terminal time such that the final state is not exponentially smaller than the initial state. However, it is unlikely that such a dependence of q can be significantly improved, since renormalization of the state can be used to implement a procedure of postselection, which implies an unlikely consequence BQP = PP. For

further discussion, see Section 8 of [10].

It is possible that variations of the Carleman linearization procedure could increase the accuracy of the result. For instance, instead of using just tensor powers of u as auxiliary variables, one could use other nonlinear functions. Several previous papers on Carleman linearization have suggested using multidimensional orthogonal polynomials [6, 30]. They also discuss approximating higher-order terms with lower-order ones in (3.2) instead of simply dropping them, possibly improving accuracy. Such changes would however break the upper triangularity of the resulting linear ODE, which could affect the quantum implementation.

The quantum part of the algorithm might also be improved. In this paper we limit ourselves to the first-order Euler method to discretize the linearized ODEs in time. This is crucial for the analysis in Lemma 3, which states the global error increases at most linearly with T. To establish this result for the Euler method, it suffices to choose the time step (4.23) to ensure $\|I + Ah\| \leq 1$, and then estimate the growth of global error by (4.46). However, it is unclear how to give a similar bound for higher-order numerical schemes. If this obstacle could be overcome, the error dependence of the complexity might be improved.

It is also natural to ask whether our approach can be improved by taking features of particular systems into account. Since the Carleman method has only received limited attention and has generally been used for purposes other than numerical integration, it seems likely that such improvements are possible. In fact, the numerical results discussed in Section 6 (see in particular Figure 1) suggest that the condition $\mathrm{R} < 1$ is not a strict requirement for the viscous Burgers equation, since we observe convergence even though $\mathrm{R} \approx 41$. This suggests that some property of equation (6.3) makes it more amenable to Carleman linearization than our current analysis predicts. We leave a detailed investigation of this for future work.

When contemplating applications, it should be emphasized that our approach produces a state vector that encodes the solution without specifying how information is to be extracted from that state. Simply producing a state vector is not enough for an end-to-end application since the full quantum state cannot be read out efficiently. In some cases in may be possible to extract useful information by sampling a simple observable, whereas in other cases, more sophisticated postprocessing may be required to infer a desired property of the solution. Our method does not address this issue, but can be considered as a subroutine whose output will be parsed by subsequent quantum algorithms. We hope that future work will address this issue and develop end-to-end applications of these methods.

Finally, the algorithm presented in this paper might be extended to solve related mathematical problems on quantum computers. Obvious candidates include initial value problems with time-dependent coefficients and boundary value problems. Carleman methods for such problems are explored in [37], but it is not obvious how to implement those methods in a quantum algorithm. It is also possible that suitable formulations of problems in nonlinear optimization or control could be solvable using related techniques.

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