

# Quantum algorithm for nonlinear differential equations

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*Abstract:* Quantum computers are known to provide an exponential advantage over classical computers for the solution of linear differential equations in high-dimensional spaces. Here, we present a quantum algorithm for the solution of *nonlinear* differential equations. The quantum algorithm provides an exponential advantage over classical algorithms for solving nonlinear differential equations. Potential applications include the Navier-Stokes equation, plasma hydrodynamics, epidemiology, and more.

Quantum computers have been shown to provide an exponential advantage over classical computers for the solution of linear differential equations [1-2]. Such quantum linear differential equation solvers provide a potential application for near term intermediate scale quantum computers [3]. Many useful differential equations are nonlinear. Previous efforts to develop quantum algorithms for nonlinear differential equations resulted in methods that scaled poorly: the number of resources required grew exponentially with the integration time [4]. Here, we present a quantum algorithm for the solution of nonlinear differential equations where the number of resources required grows quadratically in the integration time, and logarithmically in the dimension of the state space, thereby provid-

ing an exponential advantage over classical differential equation solvers. The nonlinear differential equation algorithm considerably expands the set of potential applications of near term quantum computers.

The basic method that we employ is to encode the vector representing the state of the system to be investigated as a quantum state. As with quantum linear differential equation solvers, the potential exponential advantage over classical computers arises because the dimension of that quantum state is exponential in the number of qubits/qudits in the state, allowing the exploration of very high-dimensional state spaces. To implement the nonlinearity, we employ multiple copies of that state, allowing the solution of differential equations whose nonlinear terms are polynomial in that state. We combine methods for simulating the dynamics of the nonlinear Schrödinger equation with quantum linear differential equation solvers to obtain an algorithm that integrates the nonlinear equation over time  $t$  using a number of additional copies that scales quadratically in  $t$  in regimes where the dynamics of the original equation is ‘reasonably’ stable – i.e., the Lyapunov exponents of the dynamics are not too large. As with classical methods for solving nonlinear equations, the accuracy of the nonlinear equation solver depends on the underlying numerical integration method used – explicit, implicit, multi-time step, etc. – and on the stability of the particular nonlinear equation to be integrated. The quadratic scaling of the algorithm presented here arises from the intrinsic error scaling of the Euler forward method and may be reduced by using more sophisticated numerical methods.

Let  $x \in \mathcal{C}^d$  be a state vector in a  $d$ -dimensional complex vector space. We want to solve equations of the form

$$\frac{dx}{dt} + f(x)x = b(t), \tag{1}$$

where  $f(x)$  is a  $d \times d$  matrix that is an order  $m$  polynomial function of the vectors  $x$  and  $x^\dagger$ . In the supplementary material, we show how to increase the dimensionality of  $x$  by 1

to write

$$f(x) = x^\dagger{}^{\otimes m} F x^{\otimes m}, \quad (2)$$

for a suitable tensor  $F$ . For the applications considered here, we assume that  $F$  is sparse and that its entries are readily computable. This will be the case, for example, for nonlinear equations which describe physical systems that are governed by local interactions.

*The nonlinear Schrödinger equation*

When  $f$  is anti-Hermitian, and there is no driving term  $b(t)$ , we can solve equations of the form (1) efficiently in the quantum mechanical setting by implementing a nonlinear Schrödinger equation [5-11]. The nonlinear Schrödinger equation for a single system arises by applying the usual linear Schrödinger evolution to multiple identical interacting copies of the original system, and taking the limit that the number of copies becomes large. See [5-11] and the Supplementary Material for details of scaling and errors for this approach: because of its nonlinearity, many questions about the domain of applicability of the nonlinear Schrödinger equation remain open.

The basic approach is as follows. Take  $n \gg m$  copies of the initial state  $x(0)^{\otimes n}$ , and apply the Hamiltonian

$$H = -i \binom{n}{m}^{-1} \sum_{j_1 \dots j_m} F_{j_1 \dots j_m}, \quad (3)$$

to the initial state  $x(0)^{\otimes n}$ . Here  $F_{j_1 \dots j_m}$  is the tensor  $F$  applied to the  $m$  distinct subsystems labeled by  $j_1 \dots j_m$ . Look at the short-time behavior of the  $n$ -system linear Schrödinger equation:

$$e^{-iH\Delta t} x^{\otimes n} = (I - iH\Delta t - (1/2)H^2\Delta t^2 + O(\Delta t^3)) x^{\otimes n}. \quad (4)$$

The short time behavior of any one of the copies is obtained by tracing out the other copies, and one obtains the effective single system dynamics

$$x \rightarrow (I - \Delta t f(x))x + O(E^2 \Delta t^2), \quad (5)$$

where  $E$  is the average value of  $|f(x)|$  over that time period. That is, the short time behavior of each copy obeys equation (1) for no driving. The deviations from the correct nonlinear behavior are suppressed by a factor of  $1/n$  in the second and higher order terms (see [5-11] and Supplementary Material).

Equation (5) shows that to integrate the nonlinear Schrödinger equation over  $T = t/\Delta t$  discretized Trotter steps, we need to take  $T$  sufficiently large that  $E^2 T \Delta t^2 = E^2 t \Delta t = E^2 t^2 / T$  is small. That is, the number of discretized steps required grows quadratically with the time over which the equation is to be integrated. In the Supplementary Material, we compare the discretized integration of the full multi-system time evolution, equation (4), with the discretized integration of the nonlinear single system evolution, equation (5), and count terms that differ between the two integrations at each order in  $t$ . We show that as long as the number of copies is significantly greater than  $T$ , the nonlinear Schrödinger equation approximation holds to accuracy  $\epsilon$  for times  $t$  such that  $E^2 t \Delta t m^2 / n < \epsilon$ . That is, the number of copies required also scales quadratically in  $t$ .

In addition, we have to pay attention to the tendency of the nonlinear Schrödinger equation to amplify small deviations in the wave function exponentially: the accuracy of the numerical integration of the nonlinear Schrödinger equation breaks down when the nonlinearity amplifies the energy  $E$  sufficiently that  $E^2 t \Delta t > O(\epsilon)$ .

The nonlinear Schrödinger equation is a mean-field equation in which the nonlinear unitary dynamics of a single system is determined by weak interactions with many other systems in the same state. Deviation from the single-system unitary nonlinear Schrödinger dynamics come from entangling terms in the dynamics, which only arise at second order in equation (4), and which are suppressed by a factor of  $1/n$ . In practice, the nonlinear Schrödinger equation can provide a highly accurate description of weakly interacting identical particles: a well-known example of its application is the Gross-Pitaevskii equation

which describes the dynamics of bosons in a Bose-Einstein condensate, where each boson can be described by the same wave function. In the case of the Gross-Pitaevskii equation, the the tensor  $F$  is sparse: its linear part is simply the single-particle Schrödinger equation, and its nonlinear part represents a spatially local interaction between the bosons.

*Quantum solution to general nonlinear differential equations*

As just seen, the case of a nonlinear equation where  $f(x)$  is anti-Hermitian is a special case that is tailor-made for solution by quantum computers: simply use standard techniques of quantum simulation to simulate the dynamics induced by the Hamiltonian of equation (3). Equation (1) without driving then takes the form of a nonlinear Schrödinger equation, and the resulting multi-system quantum dynamics goes through a series of quantum states that represent the solution to equation (1). Features of the solution can be obtained by performing quantum post-processing on the quantum states generated by the simulation [3].

It is not particularly surprising that we can use a quantum computer to solve a nonlinear Schrödinger equation: we simply perform a quantum simulation of the symmetric multi-system quantum dynamics that leads to such equations. The case of general nonlinear differential equations is harder. When  $f(x)$  is not anti-Hermitian, equation (1) encompasses a wide variety of nonlinear differential equations, including the Boltzmann equation, the Navier-Stokes equation, plasma hydrodynamics, etc. Since  $f(x)$  is not anti-Hermitian, we can't embed its solution into a nonlinear Schrödinger equation. We now show, however, that we can combine the mean-field techniques of the nonlinear Schrödinger equation with the methods of quantum linear differential equation solvers [1-2] to obtain a quantum solution to equation (1). The resulting quantum nonlinear differential equation algorithm inherits the exponential quantum advantage of the linear differential equation solvers.

*Review of quantum linear differential equation solvers*

To set up the general nonlinear case, we first review the methods by which quantum algorithms obtain a quantum representation of the solution of linear differential equations. To obtain the optimal scaling and computational complexity, sophisticated methods are required [2]. For the sake of simplicity of exposition, we review here the original method proposed in [1], mentioning extensions and elaborations as we go.

[1] showed how to map the problem of solving a general linear differential equation to that of matrix inversion, which can then be performed using the quantum linear systems algorithm [12-13]. Consider a linear differential equation of the form,

$$\frac{dx}{dt} + Ax = b(t), \quad (6)$$

where as above  $x, b \in \mathcal{C}^d$  and  $A$  is a  $d \times d$  matrix. Discretize the equation in time at intervals  $\Delta t$ , and take  $k$  to be the index for the discretized time, so that  $x_k$  and  $b_k$  are the values of  $x$  and  $b$  at time label  $k$ . We wish to integrate equation (6) numerically starting from the initial state  $x_0 \equiv b_0$ . We obtain a series of equations of the form:

$$x_0 = b_0 \quad x_1 = x_0 - \Delta t Ax_0 + \Delta t b_1 \quad \dots \quad x_{k+1} = x_k - \Delta t Ax_k + \Delta t b_k \quad \dots \quad (7)$$

Here, we have used the Euler forward method for numerical integration, but it is straightforward to implement implicit methods such as Euler backward, Crank-Nicholson, Runge-Kutta, etc., if greater numerical stability is required [3]. Written in matrix form, these equations become

$$- \begin{pmatrix} -I & 0 & 0 & \dots & 0 & 0 \\ I - \Delta t A & -I & 0 & \dots & 0 & 0 \\ 0 & I - \Delta t A & -I & \dots & 0 & 0 \\ & & \dots & & & \\ 0 & 0 & 0 & \dots & -I & 0 \\ 0 & 0 & 0 & \dots & I - \Delta t A & -I \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \dots \\ x_{T-1} \\ x_T \end{pmatrix} = \begin{pmatrix} b_0 \\ \Delta t b_1 \\ \Delta t b_2 \\ \dots \\ \Delta t b_{T-1} \\ \Delta t b_T \end{pmatrix}. \quad (8)$$

Writing in quantum form, we adjoin a time-step register  $|k\rangle$ , and encode the solution vector whose components are the  $x_k$ , and the initial state/driving vector whose components are  $b_k$ , as (unnormalized) quantum ‘history states’

$$|X\rangle = \sum_k |x_k\rangle|k\rangle, \quad |B\rangle = |b_0\rangle|k=0\rangle + \Delta t \sum_{k=1}^T |b_k\rangle|k\rangle. \quad (9)$$

Here  $|x_k\rangle$  is the quantum state corresponding to the solution to equation (8) at time-step  $k$ . When  $b_j = 0$  for  $j > 0$  (no driving), the states  $|x_k\rangle$  take the form

$$|x_k\rangle = \sum_k (I - \Delta t A)^k |x_0\rangle \approx \sum_k e^{-k\Delta t A} |x_0\rangle. \quad (10)$$

The matrix in equation (8) can be written in quantum form as

$$\mathcal{M} = \sum_{k=0}^T I \otimes |k\rangle\langle k| - \sum_{k=0}^{T-1} (I - \Delta t A) \otimes |k+1\rangle\langle k|. \quad (11)$$

We now use the quantum linear system algorithm [12-13] to solve the equation

$$\mathcal{M}|X\rangle = |B\rangle. \quad (12)$$

The quantum algorithm takes as inputs the matrix  $A$ , the initial state, and the vector of driving terms, and returns as its solution the history state  $|X\rangle$ , revealing its normalization in the process. This quantum history state can now be measured and undergo quantum post-processing [3] to reveal features of the solution to equation (6).

### *Non-Hermitian nonlinear Schrödinger equation*

We now combine quantum linear differential equation solvers with techniques borrowed from the treatment of the nonlinear Schrödinger equation [5-11] to implement a quantum nonlinear differential equation solver. The central insight is that the construction of the quantum history state equation (9) via the quantum linear differential equation solver does not require the matrix  $A$  to be Hermitian. So we simply implement a quantum version

of the dynamics of equation (1) by constructing a nonlinear Schrödinger equation for a non-Hermitian Hamiltonian, and use the quantum linear systems algorithm to construct the desired history state that corresponds to the solution of equation (1). Just as in the Hermitian case, the non-Hermitian case preserves the tensor product form of the solution state up to terms which are suppressed by  $O(1/n)$ .

Explicitly: Take  $n$  copies of the input/driving state and construct the state

$$|B\rangle^{(n)} \equiv |b_0\rangle^{\otimes n} |k=0\rangle + \Delta t \sum_{k=1}^T |b_k\rangle^{\otimes n} |k\rangle. \quad (13)$$

For the matrix  $\mathcal{M}$ , we use the operator

$$\mathcal{M}^{(n)} \equiv \sum_{k=0}^T I \otimes |k\rangle\langle k| - \sum_{k=0}^{T-1} (I - \Delta t \binom{n}{m}^{-1} \sum_{j_1 \dots j_m} F_{j_1 \dots j_m}) \otimes |k+1\rangle\langle k|. \quad (14)$$

The solution to the equation

$$\mathcal{M}^{(n)} |X\rangle^{(n)} = |B\rangle^{(n)}, \quad (15)$$

then takes the form of the history state

$$|X\rangle^{(n)} \approx \sum_k |x_k^1\rangle \otimes \dots \otimes |x_k^n\rangle |k\rangle, \quad (16)$$

where

$$|x_k\rangle = (I - \Delta t f(x_{k-1})) |x_{k-1}\rangle + \Delta t |b_k\rangle. \quad (17)$$

$|x_k\rangle$  is the Euler forward solution to equation (1) at time  $k$ , and  $|x_k^j\rangle = |x_k\rangle$  for all  $j$ . The approximation sign in equation (16) comes from the error induced by the nonlinear Schrödinger equation approximation (see Supplementary Material), and means that the marginal density matrices for each subsystem at time-step  $k$  are equal to

$$|x_k\rangle\langle x_k| + O(|E|^2 T m^2 \Delta t^2 / n). \quad (18)$$

Here,  $|E|^2$  is the average modulus squared for the complex eigenvalues of the non-Hermitian Hamiltonian  $f(x)$  over the integration time. That is, the  $n$ -fold tensor product history

state is, to lowest order in  $1/n$ , the superposition of the  $n$ -fold tensor product of the solutions to the desired nonlinear equation (1), at different times. As with the Hermitian nonlinear Schrödinger equation, the error is dominated by discretization error of the numerical integration method, and we require that

$$|E|^2 t \Delta t < O(\epsilon) \quad (19)$$

throughout the integration time  $t$ . As shown in the supplementary material, the error due to the nonlinear Schrödinger equation approximation is  $|E|^2 t \Delta t m^2 / n$ , which is suppressed by a factor of  $n$  compared with the error of the numerical integration.

*Errors, stability, and range of applicability*

For ease of explication, we have presented the simplest possible version of the quantum nonlinear differential equation algorithm, which is based on lowest order Trotterization [14]. Higher order Trotterization, and various other techniques such as the use of higher order implicit numerical methods, are likely to improve the error scaling. The approximations leading to the quantum solution, equation (17) of the nonlinear equation (1), must necessarily break down if the nonlinearity leads to large exponential growth, e.g., if the nonlinear equation has positive Lyapunov exponents, and we try to integrate for times sufficiently long that equation (19) is violated. Indeed, if they did not break down, one could use the method to amplify exponentially small differences in the initial wave function, which would allow the solution of NP-complete problems on a quantum computer [15-17]! Note that this issue also arises for the quantum solution of linear equations when the governing matrix has eigenvalues with positive real part. In both the linear and the nonlinear cases, the quantum solution gives an exponential speed up only over times where such amplification does not result in violations of equation (19).

The long time accuracy of the quantum nonlinear equation solver in the presence of

positive Lyapunov exponents can be better than that of the quantum linear equation solver with positive real eigenvalues, because the nonlinearity implies that the directions in which the solution is exponentially expanding change over time, so that there is no net exponential growth: this is the case, for example, when the sum of the Lyapunov exponents is negative, leading to fractal solutions such as strange attractors. In the nonlinear case, because of its construction, the quantum solver works best when applied to problems where the mean-field approximation remains valid throughout the period of the time evolution that is investigated, and where the underlying numerical method (Euler forward, Euler backward, Crank-Nicholson, Runge-Kutta, etc.) is accurate. The performance of the quantum nonlinear differential equation algorithm will vary depending on the characteristics – notably, the stability – of the nonlinear equation that it is given to solve.

*Applications:*

To obtain the exponential speed-up afforded by the quantum nonlinear differential equation algorithm, the tensor  $F$  must be sparse and have computable entries. In the case of the Boltzmann equation, for example, the vector  $x$  represents a vector of probabilities or densities  $p(y, v)$  in the single-particle phase space of positions  $y$  and velocities  $v$ , the linear part of the tensor  $F$  represents the noninteracting diffusive dynamics of the fluid, and the nonlinear part of  $F$  represents a spatially local, momentum conserving scattering dynamics. The spatial locality of the interaction, combined with the well-specified form of the scattering interaction, implies that  $F$  is sparse and that its entries are computable. Similarly, in the Navier-Stokes equation, the locality of interactions combined with momentum conservation gives rise to sparse, computable tensors  $F$ . The nonlinear equations for plasma hydrodynamics include both the dynamics of charged particles and of the electromagnetic fields: although these equations are more complex, because

they involve local interactions and dynamics that obey the laws of physics the resulting nonlinear terms in  $F$  are again sparse and computable.

*Comparison to related work:*

The first effort to construct a quantum algorithm for nonlinear differential equations [4] also used multiple copies of the system to induce the nonlinearity, but because the algorithm required that  $m$  copies at each Trotter step be sacrificed to produce a single copy at the next step, the resources required by the algorithm scaled as exponentially in the number of Trotter steps  $T$ : for pairwise nonlinear interactions, for example, the resources required in [4] scale as  $2^T$ . The nonlinear, non-Hermitian Schrödinger equation approach presented here obtains quadratic scaling in  $T$  by retaining all copies of the system at each step, rather than discarding them. Other efforts to present quantum algorithms for nonlinear differential equations rely on variational techniques [18], but do not supply the provable exponential speed-up given by our algorithm. A separate method involves the Madelung hydrodynamic approach to quantum mechanics [19]. A recently posted work [20] presents a method similar to that pursued here, using a linear system over multiple copies to induce the single-system nonlinearity and applying the quantum linear differential equation solver, but uses classical Carleman linearization instead of the quantum nonlinear Schrödinger linearization technique.

*Conclusion:*

This paper showed that quantum computers can in principle attain an exponential advantage over classical computers for solving nonlinear differential equations. The main potential advantage of the quantum nonlinear equation algorithm over classical algorithms is that it scales logarithmically in the dimension of the solution space, making it a natural candidate for applying to high dimensional problems such as the Navier-Stokes

equation and other nonlinear fluids, plasmas, etc. The method developed here could be applied to continuous variable quantum systems using the techniques of [21]. Like quantum linear differential equation solvers, the quantum nonlinear solver presents its solution as a quantum history state, equation (9), which allows quantum post-processing [3] to extract features such as the power spectrum (quantum Fourier transform), principal components (quantum singular value transformation), and multiscale behavior (quantum wavelet transforms).

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*Supplementary material:*

We show here how to write an arbitrary  $m$ 'th order polynomial in the form of equation (2):

$$f(x) = x^\dagger{}^{\otimes m} F x^{\otimes m}. \quad (S1)$$

Let  $x$  be  $d$ -dimensional:

$$x = (x_1, \dots, x_d)^\top. \quad (S2a)$$

Add an extra dimension:

$$\tilde{x} = (x_0, x_1, \dots, x_d)^\top, \quad (S2b)$$

and augment the original differential equation (1) with the initial condition  $x_0 = 1$ , and  $dx_0/dt = 0$ , so that  $x_0$  remains 1 at all times.

The addition of an extra dimension whose coefficient takes a constant value allows us to include arbitrary polynomials in the  $\{x_j, \bar{x}_j\}$  in  $f(x)$ . For example, in the case  $m = 2$ , to include the monomial  $\bar{x}_1^2 x_2$  in the  $|2\rangle\langle 1|$  entry of  $f(x)$  (here we use quantum notation), we include in  $F$  a term

$$G_{\bar{1}\bar{1}2} = |2\rangle\langle 1| \otimes (1/2)(|1\rangle_1\langle 2| \otimes |1\rangle_2\langle 0| + |1\rangle_2\langle 0| \otimes |1\rangle_2\langle 2|). \quad (S3)$$

Performing the inner products yields:

$$\langle x|\langle x|G_{\bar{1}\bar{1}2}|x\rangle|x\rangle = \bar{x}_1^2 x_2 |2\rangle\langle 1|, \quad (S4)$$

the desired result.

### *Normalization*

In the usual, Hermitian version of the nonlinear Schrödinger equation, the states  $|x_j\rangle$  automatically remain normalized to one. To maintain this normalization under the action of the non-Hermitian operator of equation (14), we rescale the variables so that

$\tilde{x}^\dagger \tilde{x} \leq 1/2$  for the period over which we integrate the equation, and add an additional dimension whose real coefficient  $x_{d+1}$  obeys  $x_{d+1}^2 = 1 - \tilde{x}^\dagger \tilde{x}$ . To maintain the normalization throughout the integration period,  $x_{d+1}$  is taken to obey the nonlinear equation

$$\frac{dx_{d+1}}{dt} = \frac{d}{dt}(1 - \tilde{x}^\dagger \tilde{x})^{1/2} = -(1/2)(1 - \tilde{x}^\dagger \tilde{x})^{-1/2} \left( \frac{d\tilde{x}^\dagger}{dt} \tilde{x} + \tilde{x}^\dagger \frac{d\tilde{x}}{dt} \right) \quad (S5),$$

which we implement within our framework by expanding  $(1 - y)^{-1/2}$  as a Taylor series about the point  $y = 0$ . The Taylor series converges exponentially, and so we augment  $m$  by  $\log(1/\epsilon)$  to maintain accuracy  $\epsilon$  throughout.

#### *Accuracy of the nonlinear Schrödinger equation*

We compare the time evolution of the full multi-system linear Schrödinger equation with the nonlinear Schrödinger equation. The analysis holds for both the Hermitian and non-Hermitian case. Because the algorithm works by discretization in time, we write the discretized Schrödinger evolution for the full system as

$$e^{-iT\Delta t H} |x_0\rangle^{\otimes n} \approx (I - i\Delta t H)^T |x_0\rangle^{\otimes n} = \left( I - \Delta t \binom{n}{m}^{-1} \sum_{i_1 \dots i_m} F_{i_1 \dots i_m} \right)^T |x_0\rangle^{\otimes n}, \quad (S6)$$

and the accuracy holds to the usual first order discretized approximation [16]. We compare this with the discretized evolution of a single copy under the nonlinear Schrödinger equation:

$$(1 - f(x_{T-1})\Delta t)(1 - f(x_{T-2})\Delta t) \dots (1 - f(x_0)\Delta t) |x_0\rangle. \quad (S7)$$

Rewrite (S6) in density matrix form:

$$\left( I - \Delta t \binom{n}{m}^{-1} \sum_{i_1 \dots i_m} F_{i_1 \dots i_m} \right)^T (|x_0\rangle\langle x_0|)^{\otimes n} \left( I - \Delta t \binom{n}{m}^{-1} \sum_{i'_1 \dots i'_m} F_{i'_1 \dots i'_m}^\dagger \right)^T. \quad (S8)$$

First, look at single step:  $T = 1$ . The time evolution of the first subsystem is obtained by tracing out subsystems  $2 \dots n$  in equation (S8). The first order terms are the same as in the discretized nonlinear Schrödinger dynamics. At second order, terms where the set

of indices  $i_1 \dots i_m$  have no overlap with the set of indices  $i'_1 \dots i'_m$  give a dynamics which is exactly the discretized nonlinear Schrödinger equation dynamics:

$$|x_0\rangle\langle x_0| \rightarrow |x_1\rangle\langle x_1| = (1 - f(x_0)\Delta t)|x_0\rangle\langle x_0|(1 - f^\dagger(x_0)\Delta t). \quad (S9)$$

Second order terms that do have overlap between the two sets of indices represent potentially entangling dynamics that departs from the tensor product form of the solution. The fraction of terms in which no such overlap occurs is approximately equal to, and bounded above by

$$(1 - m/n)^m \approx 1 - m^2/n. \quad (S10)$$

This approximation holds as long as  $n \gg mT$ : that is, the number of copies must be larger than the number of time steps  $T$ . That is, the state produced by the full dynamics (S6) yields an overall state that is approximately the desired state,

$$|x_1\rangle\langle x_1|^{\otimes n}, \quad (S11)$$

with single-system density matrices that are of the form

$$|x_1\rangle\langle x_1| + O(|E|^2\Delta t^2 m^2/n), \quad (S12)$$

where we have included the maximum energy and time scales to give the size of the deviation from the correct nonlinear Schrödinger dynamics at that step. Because the errors are generated by entangling each subsystem with all other subsystems, the error terms for any subset of subsystems manifest themselves as mixed states, so the state of any subset of  $m$  subsystems together with the errors is the state

$$(1 - \epsilon_1)|x_1\rangle\langle x_1|^{\otimes m} + \epsilon_1\eta_1 \quad (S13)$$

where  $\epsilon_1$  is  $O(|E|^2\Delta t^2 m^2/n)$  and  $\|\eta_1\|_1 \leq 3$ . For the  $m$ -fold density matrix, the fraction of terms in (S8) that give departures from the tensor product form is approximately  $2m^2/n$ ,

because we have to also count terms that entangle the  $m$  subsystems with each other. This form for the  $m$ -fold reduced density matrices means that at the next step, the errors induced by the presence of entanglement and correlations are additive.

Now apply the second time step: exactly the same argument applies, and we obtain a state whose single-system reduced density matrices are of the form

$$|x_2\rangle\langle x_2| + O(2\overline{|E|^2}\Delta t^2 m^2/n). \quad (S14)$$

where  $\overline{|E|^2}$  is the average modulus squared energy scale over the two steps (the energy scale at the second step can differ from the first). The  $m$  system reduced density matrices, for the same reason as before, are of the form

$$(1 - \epsilon_2)|x_2\rangle\langle x_2|^{\otimes m} + \epsilon_2\eta_2, \quad (S15)$$

where  $\epsilon_2$  is  $O(2\overline{|E|^2}\Delta t^2 m^2/n)$  and  $\|\eta_2\|_1 \leq 3$ . Continuing for  $T$  steps yields a final state whose single-system density matrices are of the form

$$|x_T\rangle\langle x_T| + O(\overline{|E|^2}T\Delta t^2 m^2/n). \quad (S16)$$

That is, for small  $m$  the errors introduced by the nonlinear Schrödinger equation approximation are suppressed by a factor of  $n$  compared with the discretization errors for the first order Trotterization of the Euler forward method.